Optimal tests for the two-sample spherical location problem

Christophe Ley, Yvik Swan, Thomas Verdebout

Lille 1 I Lille 2 I Lille 3 I

Document de travail

[2012–16]

"Optimal tests for the two-sample spherical location problem"

Christophe Ley, Yvik Swan and Thomas Verdebout

Université Lille Nord de France

Pôle de Recherche et d'Enseignement Supérieur

Université Lille 1 Sciences et Technologies

Université Lille 2 Droit et Santé

Lille 3 Université Sciences humaines et sociales Charles-de-Gaulle

“Optimal tests for the two-sample spherical location problem”

Christophe Ley, Yvik Swan and Thomas Verdebout

Christophe Ley
Département de Mathématique and ECARES
Université Libre de Bruxelles
Boulevard du Triomphe, CP 210
B-1050 Bruxelles, Belgium

Yvik Swan
Faculté des Sciences, de la Technologie et de la Communication
Unité de Recherche en Mathématiques
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg, Grand Duché de Luxembourg

Thomas Verdebout
PRES Université Lille Nord de France, Université Lille 3, Laboratoire EQUIPPE
EA 4018, Villeneuve d’Ascq, France.
thomas.verdebout@univ-lille3.fr

Optimal tests for the two-sample spherical location problem

Christophe Ley*, Yvik Swan** and Thomas Verdebout†

*Département de Mathématique and ECARES
Université Libre de Bruxelles
Boulevard du Triomphe, CP 210
B-1050 Bruxelles, Belgium

** Faculté des Sciences, de la Technologie et de la Communication
Unité de Recherche en Mathématiques
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg, Grand Duché de Luxembourg

† EQUIPPE
Université Lille III
Domaine Universitaire du Pont de Bois, BP 60149
F-59653 Villeneuve d'Ascq Cedex, France

Abstract

We tackle the classical two-sample spherical location problem for directional data by having recourse to the Le Cam methodology, habitually used in classical "linear" multivariate analysis. More precisely we construct locally and asymptotically optimal (in the maximin sense) parametric tests, which we then turn into semi-parametric ones in two distinct ways. First, by using a studentization argument; this leads to so-called pseudo-FvML tests. Second, by resorting to the invariance principle; this leads to efficient rank-based tests. Within each construction, the semi-parametric tests inherit optimality under a given distribution (the FvML in the first case, any rotationally symmetric one in the second) from their parametric counterparts and also improve on the latter by being valid under the whole class of rotationally symmetric distributions. Asymptotic relative efficiencies are calculated and the finite-sample behavior of the proposed tests is investigated by means of a Monte Carlo simulation.

Keywords: Directional statistics, local asymptotic normality, pseudo-FvML tests, rank-based inference, two-sample spherical location problem.

2000 MSC: 62H11, 62H15, 62G10

1. Introduction

Spherical or directional data naturally arise in a broad range of earth sciences such as geology and astrophysics (see, e.g., Watson 1983 or Mardia and Jupp 2000), as well as in studies of animal behavior (see Fisher *et al.* 1987) or even in neuroscience (see Leong and Carlile 1998). Although this field of research is as old as mathematical statistics themselves and raises the same questions as the more classical "linear" statistics, its methodical and systematic study only started in the 1950s under the impetus of Fisher (1953)'s pioneering work (see Mardia and Jupp 2000). It is now common practice to view directional data as realizations of random vectors $\mathbf{X}$ taking values on the surface of the unit hypersphere $\mathcal{S}^{k-1} := \{\mathbf{v} \in \mathbb{R}^k : \mathbf{v}'\mathbf{v} = 1\}$, $k \geq 2$, the distribution of $\mathbf{X}$ depending only on its "angular distance" from a fixed point $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ (thus such spherical distributions belong to the category of *statistical group models* and enjoy the nice properties of that class of distributions; see Chang 2004 for details). This parameter $\boldsymbol{\theta}$, which can be viewed as a "north pole" (or "mean direction") for the problem under study, then is to be considered as a spherical location parameter.

In this paper, we investigate the two-sample spherical location testing problem $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$, where $\boldsymbol{\theta}_1 \in \mathcal{S}^{k-1}$ and $\boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}$ are the respective spherical location parameters of two independent samples of i.i.d. observations with respective common distributions P_1 and P_2, say. Motivated by the fold test problem in palaeomagnetism (see McFadden and Jones 1981 and references therein), this problem has been extensively studied in the literature. Due to the difficulty of the task, most methods are either of parametric nature or restricted to small dimensions, or suffer from computational difficulties/slowness such as Wellner (1979)'s permutation test or Beran and Fisher (1998)'s bootstrap test. We refer the reader to the introduction section in Tsai (2009) for a more complete description of the strengths and flaws of the different proposals. The paper Tsai (2009) itself proposes a rank-based test for the two-sample problem. However, Tsai (2009) considers the very restrictive case where the two samples share a common distribution, that is, $P_1 = P_2$; this is particularly uncomfortable when dealing with *Fisher-von Mises-Langevin* (FvML hereafter) distributions, where the additional concentration parameter $\kappa > 0$ then needs to be the same for both samples in order for the test to be valid. Thus, to the best of the authors' knowledge, the only computationally simple, efficient and asymptotically distribution-free test for the general null hypothesis $\mathcal{H}_0$ above is the *pseudo-FvML* test given in Watson (1983). The idea behind his test has the same flavor as the *pseudo-Gaussian* tests in the classical "linear" framework (see, for instance, Muirhead and Waternaux 1980 or Hallin and Paindaveine 2008 for more information on pseudo-Gaussian procedures). More concretely, since the FvML distribution is considered as the spherical analogue of the Gaussian distribution (see Section 2 for an explanation of this fact), Watson has chosen the FvML as basis distribution and hence constructed his pseudo-FvML tests by "correcting" the FvML-likelihood ratio test, optimal under a couple (P_1, P_2) of FvML distributions with fixed concentration parameters κ_1 and κ_2, in such a way that the resulting test remains valid under a large class of distributions.

Our aim in the present paper consists in proposing new tests for the two-sample spherical location problem. More concretely, we aim to construct tests that are optimal (in the maximin sense) under a given pair of distributions (P_1, P_2) but remain valid (in the sense that they meet the nominal level constraint) under a broad class of distributions, namely the rotationally symmetric distributions (introduced by Saw 1978; see Section 2 below for a definition). The backbone of our approach is the so-called Le Cam methodology (see Le Cam 1986), as adapted to the spherical setup by Ley *et al.* (2012). Of utmost importance for our aims here is the *uniform local asymptotic normality* (ULAN) of a sequence of rotationally symmetric distributions established therein. This property allows us to construct optimal parametric tests under the pair (P_1, P_2). This optimality, however, is thwarted by the non-validity of the tests under any pair (Q_1, Q_2) distinct from (P_1, P_2). In order to palliate this problem, we have recourse to two classical tools: a *studentization* argument, which eventually leads to Watson (1983)'s pseudo-FvML tests, and the *invariance principle*, yielding optimal rank-based tests. Both tests are of semi-parametric nature.

The paper is organized as follows. In Section 2, we collect the main assumptions of the paper, summarize asymptotic results in the context of rotationally symmetric distributions and show how to construct the announced optimal parametric tests for the two-sample spherical location problem. We then extend the latter to pseudo-FvML tests in Section 3 and to rank-based tests in Section 4, and study their respective asymptotic behavior in each section. Asymptotic relative efficiencies are provided in Section 4. The theoretical results are corroborated via a Monte Carlo simulation in Section 5. Finally an appendix collects the proofs.

2. Main assumptions, notations and important preliminary results

2.1. Rotational symmetry and the particular rôle of the FvML distribution

Throughout, the two samples of data points $\mathbf{X}_{11}, \ldots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \ldots, \mathbf{X}_{2n_2}$ are assumed to belong to the unit sphere $\mathcal{S}^{k-1}$ of $\mathbb{R}^k$, $k \geq 2$, and to satisfy

Assumption A. (Rotational symmetry) $\mathbf{X}_{11}, \ldots, \mathbf{X}_{1n_1}$ (resp., $\mathbf{X}_{21}, \ldots, \mathbf{X}_{2n_2}$) are i.i.d. with common distribution $\mathrm{P}_{\boldsymbol{\theta};f_1}$ (resp., $\mathrm{P}_{\boldsymbol{\theta};f_2}$) characterized by a density (with respect to the usual surface area measure on spheres)

$$\mathbf{x} \mapsto c_{k,f_i}\; f_i(\mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{k-1}, \tag{2.1}$$

where $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ is a location parameter and $f_i : [-1, 1] \to \mathbb{R}_0^+$ is absolutely continuous and (strictly) monotone increasing, $i = 1, 2$. Then, if $\mathbf{X}$ has density (2.1), the density of $\mathbf{X}'\boldsymbol{\theta}$ is of the form

$$t \mapsto \tilde{f}_i(t) := \frac{\omega_k\; c_{k,f_i}}{B(\frac{1}{2}, \frac{1}{2}(k-1))} f_i(t)(1-t^2)^{(k-3)/2}, \quad -1 \leq t \leq 1,$$

where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of $\mathcal{S}^{k-1}$ and $B(\cdot, \cdot)$ is the beta function. The corresponding cumulative distribution function (cdf) is denoted by $\tilde{F}_i(t)$, $i = 1, 2$.

The f_i's are called *angular functions* (because the distribution of each $\mathbf{X}_{ij}$ depends only on the angle between it and the location $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$). Throughout the rest of this paper, we denote by $\mathcal{F}^2$ the collection of pairs of angular functions $\underline{f} := (f_1, f_2)$. Although not necessary for the definition to make sense, monotonicity of f_i ensures that surface areas in the vicinity of the location parameter $\boldsymbol{\theta}$ are allocated a higher probability mass than more remote regions of the sphere. This property happens to be very appealing from the modelling point of view. Rotationally symmetric distributions also enjoy some appealing stochastic properties. Indeed, as shown in Watson (1983), for a random vector $\mathbf{X}$ distributed according to some $\mathrm{P}_{\boldsymbol{\theta};f_i}$ as in Assumption A, not only is the multivariate sign vector $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}) := (\mathbf{X} - (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta})/||\mathbf{X} - (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta}||$ uniformly distributed on $\mathcal{S}^{\boldsymbol{\theta}^\perp} := \{\mathbf{v} \in \mathbb{R}^k \,|\, \|\mathbf{v}\| = 1, \mathbf{v}'\boldsymbol{\theta} = 0\}$ but also the angular distance $\mathbf{X}'\boldsymbol{\theta}$ and the sign vector $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$ are stochastically independent.

The class of rotationally symmetric distributions contains several spherical distributions such as the linear, the logarithmic, the logistic (all three are provided in Section 4 below) or the wrapped normal distribution; for other examples and a more detailed description of the aforementioned ones, we refer to Duerinckx and Ley (2012). By far the most popular and most used rotationally symmetric distribution is the FvML distribution (named, according to Watson 1983, after von Mises 1918, Fisher 1953, and Langevin 1905), whose density is of the form

$$f_{\mathrm{FvML}(\kappa)}(\mathbf{x};\boldsymbol{\theta}) = C_k(\kappa)\exp(\kappa\mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{k-1},$$

where $\kappa > 0$ is a concentration or dispersion parameter, $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ a location parameter and the normalization constant $C_k(\kappa)$ is equal to

$$C_k(\kappa) = \frac{\kappa^{k/2-1}}{(2\pi)^{k/2} I_{k/2-1}(\kappa)},$$

with $I_{k/2-1}(\kappa)$ the modified Bessel function of the first kind and of order $k/2 - 1$. In what follows, we shall replace $f_{\mathrm{FvML}(\kappa)}$ with the lighter notation ϕ_κ. As already mentioned in the Introduction, the FvML is considered as the spherical analogue of the Gaussian distribution for purposes of mathematical statistics (see Schaeben 1992 for a discussion on analogues of the Gaussian distribution). This analogy is mainly due to the fact that the FvML distribution can be characterized by the empirical spherical mean $\hat{\boldsymbol{\theta}}_{\mathrm{Mean}} := \sum_{i=1}^n \mathbf{X}_i/||\sum_{i=1}^n \mathbf{X}_i||$, $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathcal{S}^{k-1}$, as the Maximum Likelihood Estimator (MLE) of its spherical location parameter, similarly as the Gaussian distribution can be characterized by the empirical mean $n^{-1}\sum_{i=1}^n \mathbf{X}_i$, $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^k$, as the MLE of its classical (linear) location parameter. More precisely, it has been shown that the Gaussian distribution is the only absolutely continuous (univariate and multivariate) distribution for which the sample mean is for all samples of fixed sample size $n \geq 3$ the MLE of the location parameter (see Azzalini and Genton 2007 for the latest version of this result first expressed in Gauss 1809), and exactly the same result is known to hold true for the FvML distribution among spherical distributions (see Duerinckx and Ley 2012, where the earlier findings of von Mises 1918 for dimension $k = 2$, Arnold 1941 and Breitenberger 1963 for dimension $k = 3$ and Bingham and Mardia 1975 for any dimension have been generalized to yield the latter statement)[1].

[1]It is interesting in this context to note that Gauss, in his manuscript "Theoria motus corporum coelestium in sectionibus conicis solem ambientium" of 1809, has defined the famous distribution named after him by searching for the probability law for which the sample mean is always the MLE of the location parameter, and that von Mises, in 1918, aiming at constructing a circular analogue of the Gaussian distribution, started precisely from this classical MLE characterization.

2.2. Le Cam optimal parametric tests for the two-sample spherical location problem

As stated in the Introduction, our first objective is to construct locally and asymptotically optimal parametric tests by having recourse to the Le Cam methodology. The main ingredient for this construction rests on the ULAN property of the parametric model $\left(\left\{\mathrm{P}^{(n)}_{\boldsymbol{\theta}_1;f_1} \mid \boldsymbol{\theta}_1 \in \mathcal{S}^{k-1}\right\}, \left\{\mathrm{P}^{(n)}_{\boldsymbol{\theta}_2;f_2} \mid \boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}\right\}\right)$ for a fixed pair of angular functions (f_1, f_2), where $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_i;f_i}$ stands for the joint distribution of $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i}$, $i = 1, 2$. We further denote by $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}}$ the joint law combining $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_1;f_1}$ and $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_2;f_2}$. In order to be able to state our results, we need to impose a certain control on the respective sample sizes n_1 and n_2, which will be achieved via the following

Assumption B. Letting $n = n_1 + n_2$, both $r_1^{(n)} := n_1/n$ and $r_2^{(n)} := n_2/n$ converge to finite constants r_1 and r_2 respectively as $n \to \infty$.

This condition explains why, in what precedes and in what follows, we simply use the superscript $^{(n)}$ for the different quantities at play and do not specify whether they are associated with n_1 or n_2.

Informally, a sequence of rotationally symmetric models $\left\{\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}} \mid \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}\right\}$ is ULAN if the logarithm of the likelihood ratio $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1^{(n)}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}_2^{(n)}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{f}}/\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1^{(n)},\boldsymbol{\theta}_2^{(n)});\underline{f}}$ allows a specific form of (probabilistic) Taylor expansion, with $\boldsymbol{\theta}_1^{(n)}, \boldsymbol{\theta}_2^{(n)} \in \mathcal{S}^{k-1}$ such that $\boldsymbol{\theta}_i^{(n)} - \boldsymbol{\theta}_i = O(n^{-1/2})$, $i = 1, 2$, and $\mathbf{t}_1^{(n)}, \mathbf{t}_2^{(n)} \in \mathbb{R}^k$ bounded sequences of perturbations such that $\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)}$ remains on the unit sphere for $i = 1, 2$. The latter condition means that each $\mathbf{t}_i^{(n)}$ needs to satisfy

$$\begin{aligned} 0 &= (\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)})'(\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)}) - 1 \\ &= 2n_i^{-1/2}(\boldsymbol{\theta}_i^{(n)})'\mathbf{t}_i^{(n)} + n_i^{-1}(\mathbf{t}_i^{(n)})'\mathbf{t}_i^{(n)}. \end{aligned} \tag{2.2}$$

Consequently, $\mathbf{t}_i^{(n)}$ must be such that $2n_i^{-1/2}\,(\boldsymbol{\theta}_i^{(n)})'\mathbf{t}_i^{(n)} + n_i^{-1}(\mathbf{t}_i^{(n)})'\mathbf{t}_i^{(n)} = 0$ or, equivalently, such that $2n_i^{-1/2}(\boldsymbol{\theta}_i^{(n)})'\mathbf{t}_i^{(n)} + o(n_i^{-1/2}) = 0$. In other words, for $\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)}$ to remain in $\mathcal{S}^{k-1}$, $\mathbf{t}_i^{(n)}$ must belong, up to a $o(n_i^{-1/2})$ quantity, to the tangent space to $\mathcal{S}^{k-1}$ at $\boldsymbol{\theta}_i^{(n)}$.

Now, $\mathcal{S}^{k-1}$ is a non-linear manifold and as a consequence, establishing the ULAN property of a sequence of rotationally symmetric models is all but easy. Ley *et al.* (2012) handle this difficulty by resorting to a natural re-parameterization in terms of spherical coordinates, for which it is possible to prove ULAN, subject to the following necessary condition.

Assumption C. The Fisher information associated with the spherical location parameter is finite; this finiteness is ensured if, for $i = 1, 2$ and letting $\varphi_{f_i} := \dot{f}_i/f_i$ ($\dot{f}_i$ is the a.e.-derivative of f_i), $\mathcal{J}_k(f_i) := \int_{-1}^{1} \varphi_{f_i}^2(t)(1-t^2)\tilde{f}_i(t)dt < +\infty$.

After obtaining the ULAN property for this new parameterization, Ley *et al.* (2012) use a lemma from Hallin *et al.* (2010) explaining how to transpose ULAN from one parameterization to another. This finally yields in the property given in Proposition 2.2 of Ley *et al.* (2012). Here, we obviously need a slightly different version of that result since we are dealing with the two-sample problem. Fortunately, the inner-sample independence and the mutual independence between the two samples entails that we can readily write down the ULAN property in the present setup.

Proposition 2.1 *[ULAN for the two-sample problem] Let Assumptions A, B and C hold. Then the model* $\left\{\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}} \mid \boldsymbol{\theta}_1,\boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}\right\}$ *is ULAN with central sequence* $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}} := \left((\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_1;f_1})',(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_2;f_2})'\right)'$, *where*

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_i;f_i} := n_i^{-1/2}\sum_{j=1}^{n_i}\varphi_{f_i}(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)(1-(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)^2)^{1/2}\mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij}), \quad i=1,2,$$

and Fisher information matrix $\boldsymbol{\Gamma}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}} := \mathrm{diag}(\boldsymbol{\Gamma}_{\boldsymbol{\theta}_1;f_1},\boldsymbol{\Gamma}_{\boldsymbol{\theta}_2;f_2})$ *where*

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta}_i;f_i} := \frac{\mathcal{J}_k(f_i)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i), \quad i=1,2.$$

More precisely, for any $\boldsymbol{\theta}_1,\boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}$ *and any bounded sequences* $\mathbf{t}^{(n)}_1,\mathbf{t}^{(n)}_2$ *as in (2.2), we have*

$$\log\left(\frac{\mathrm{P}^{(n)}_{(\boldsymbol{\theta}^{(n)}_1+n_1^{-1/2}\mathbf{t}^{(n)}_1,\boldsymbol{\theta}^{(n)}_2+n_2^{-1/2}\mathbf{t}^{(n)}_2);\underline{f}}}{\mathrm{P}^{(n)}_{(\boldsymbol{\theta}^{(n)}_1,\boldsymbol{\theta}^{(n)}_2);\underline{f}}}\right) = (\mathbf{t}^{(n)})'\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}^{(n)}_1,\boldsymbol{\theta}^{(n)}_2);\underline{f}} - \frac{1}{2}(\mathbf{t}^{(n)})'\boldsymbol{\Gamma}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}}\mathbf{t}^{(n)} + o_{\mathrm{P}}(1),$$

where $\mathbf{t}^{(n)} := ((\mathbf{t}^{(n)}_1)',(\mathbf{t}^{(n)}_2)')'$, *and* $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}^{(n)}_1,\boldsymbol{\theta}^{(n)}_2);\underline{f}} \xrightarrow{\mathcal{L}} \mathcal{N}_{2(k-1)}(\mathbf{0},\boldsymbol{\Gamma}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}})$, *both under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}}$, *as* $n\to\infty$.

Proposition 2.1 is a straightforward consequence of the result in Ley *et al.* (2012), and hence the proof is omitted. With this in hand, constructing optimal $\underline{f}$-parametric procedures (that is, under the pair of densities with respective specified angular functions f_1 and f_2) for testing $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ is plain sailing. Indeed, denoting by $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ the common null hypothesis value of the location parameter, the corresponding parametric test statistic is given by

$$(\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}})'(\boldsymbol{\Gamma}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}})^{-1}\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}}.$$

Evidently, the unknown value $\boldsymbol{\theta}$ needs to be estimated, but we leave that issue to the subsequent sections. Intuitively, this construction of optimal parametric tests follows from the fact that the second-order expansion of the log-likelihood ratio for the model $\left\{\mathrm{P}^{(n)}_{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2);\underline{f}} \mid \boldsymbol{\theta}_1,\boldsymbol{\theta}_2 \in \mathcal{S}^{k-1}\right\}$ strongly resembles the log-likelihood ratio for the classical Gaussian shift experiment, for which optimal procedures are well-known and are based on the corresponding first-order term. For more details and more formal explanations, we refer the reader to Le Cam (1986), Section 11.9.

As mentioned in the Introduction, the optimal parametric tests suffer from the drawback of being only valid under the pair (f_1, f_2). Since it is highly unrealistic in practice to assume that the underlying densities are known, these tests are useless for practitioners. The next two sections contain two distinct solutions allowing to set this problem right.

3. Pseudo-FvML tests.

For a given pair of FvML densities $(\phi_{\kappa_1}, \phi_{\kappa_2})$ with respective concentration parameters $\kappa_1, \kappa_2 > 0$ (where we do not assume $\kappa_1 = \kappa_2$), the quantities $\varphi_{\phi_{\kappa_i}}$ reduce to the constants κ_i, $i = 1, 2$, and hence the central sequences for each sample take the form

$$\begin{aligned}
\mathbf{\Delta}^{(n)}_{\boldsymbol{\theta}_i;\phi_{\kappa_i}} &:= \kappa_i n_i^{-1/2} \sum_{j=1}^{n_i} (1 - (\mathbf{X}'_{ij}\boldsymbol{\theta}_i)^2)^{1/2} \mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij}) \\
&= \kappa_i n_i^{-1/2} \sum_{j=1}^{n_i} (\mathbf{X}_{ij} - (\mathbf{X}'_{ij}\boldsymbol{\theta}_i)\boldsymbol{\theta}_i) \\
&= \kappa_i (\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i) n_i^{-1/2} \sum_{j=1}^{n_i} \mathbf{X}_{ij} \\
&=: \kappa_i (\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i) n_i^{1/2} \bar{\mathbf{X}}_i \\
&= \kappa_i (\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i) n_i^{1/2} (\bar{\mathbf{X}}_i - \boldsymbol{\theta}_i), \quad i = 1, 2.
\end{aligned}$$

Optimal FvML-based procedures for the two-sample spherical location problem are then built upon $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});(\phi_{\kappa_1},\phi_{\kappa_2})}$. We here again draw the reader's attention to the fact that this parametric test is only valid under the pair $(\phi_{\kappa_1}, \phi_{\kappa_2})$ and becomes non-valid even if only the concentration parameters change. In this section, this non-validity problem will be overcome in the following way. We will first study the asymptotic behavior of $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});(\phi_{\kappa_1},\phi_{\kappa_2})}$ under any given pair $\underline{g} = (g_1, g_2) \in \mathcal{F}^2$ and consider the newly obtained quadratic form in $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});(\phi_{\kappa_1},\phi_{\kappa_2})}$. Clearly, this quadratic form will now depend on the asymptotic variance of $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});(\phi_{\kappa_1},\phi_{\kappa_2})}$ under $\underline{g}$, hence again, for each $\underline{g}$, we are confronted to an only-for-$\underline{g}$-valid test statistic. The next and final step then consists in applying a studentization argument, meaning that we estimate this asymptotic variance quantity and study the asymptotic behavior of the new quadratic form under any pair of rotationally symmetric distributions. The final outcome of this procedure will be tests which happen to be optimal under *any* pair of FvML distributions (that is, for any values $\kappa_1, \kappa_2 > 0$) and valid under the entire class of rotationally symmetric distributions; these tests are our so-called pseudo-FvML tests.

For the sake of readability, we adopt the notation $\underline{\phi}$ for any pair $(\phi_{\kappa_1}, \phi_{\kappa_2})$ and $\mathrm{E}_f[\cdot]$ for expectation under the angular function f. The following result characterizes, for a given pair of angular functions $\underline{g} \in \mathcal{F}^2$, the asymptotic properties of the FvML-based central sequence $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}}$, both under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ and $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$ with $\mathbf{t}_1^{(n)}$ and $\mathbf{t}_2^{(n)}$ as in (2.2) for each sample.

Proposition 3.1 *Let Assumptions A, B and C hold. Then, letting $B_{k,g_i} := 1 - \mathrm{E}_{g_i}\left[(\mathbf{X}'_{ij}\boldsymbol{\theta})^2\right]$ for $i = 1, 2$, we have that $\mathbf{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}}$ is*

(i) asymptotically normal under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ with mean zero and covariance matrix

$$\mathbf{\Gamma}^*_{\boldsymbol{\theta};\underline{g}} := \mathrm{diag}\left(\mathbf{\Gamma}^*_{\boldsymbol{\theta};g_1}, \mathbf{\Gamma}^*_{\boldsymbol{\theta};g_2}\right),$$

where $\boldsymbol{\Gamma}^*_{\boldsymbol{\theta};g_i} := \frac{\kappa_i^2 B_{k,g_i}}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \quad i = 1, 2;$

(ii) asymptotically normal under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)}, \boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$ *(*$\mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2^{(n)}$ *as in (2.2)) with mean* $\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}}\mathbf{t}$ *(*$\mathbf{t} := (\mathbf{t}_1', \mathbf{t}_2')'$ *with* $\mathbf{t}_1 := \lim_{n\to\infty} \mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2 := \lim_{n\to\infty} \mathbf{t}_2^{(n)}$*) and covariance matrix* $\boldsymbol{\Gamma}^*_{\boldsymbol{\theta};\underline{g}}$*, where, putting* $C_{k,g_i} := \mathrm{E}_{g_i}[(1 - (\mathbf{X}_{ij}'\boldsymbol{\theta})^2)\varphi_{g_i}(\mathbf{X}_{ij}'\boldsymbol{\theta})]$ *for* $i = 1, 2$,

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} := \mathrm{diag}\left(\boldsymbol{\Gamma}_{\boldsymbol{\theta};\phi_{\kappa_1},g_1}, \boldsymbol{\Gamma}_{\boldsymbol{\theta};\phi_{\kappa_2},g_2}\right)$$

with $\boldsymbol{\Gamma}_{\boldsymbol{\theta};\phi_{\kappa_i},g_i} := \frac{\kappa_i C_{k,g_i}}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \quad i = 1, 2.$

See the Appendix for the proof. As the null hypothesis only specifies that both spherical locations coincide, we need to estimate the unknown common value $\boldsymbol{\theta}$. Therefore, we assume in the sequel the existence of an estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ such that the following assumption holds.

Assumption D. The estimator $\hat{\boldsymbol{\theta}} \in \mathcal{S}^{k-1}$ is such that $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is $O_{\mathrm{P}}(n^{-1/2})$ under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ for any $\underline{g} \in \mathcal{F}^2$.

Typical examples of estimators satisfying Assumption D belong to the class of M-estimators (see Chang 2004) or R-estimators (see Ley *et al.* 2012). Put simply, instead of $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}}$ we have to work with $\boldsymbol{\Delta}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{\phi}}$ for some estimator $\hat{\boldsymbol{\theta}}$ satisfying Assumption D. The next crucial result explains in how far this replacement affects the asymptotic properties established in Proposition 3.1.

Proposition 3.2 *Let Assumptions A, B and C hold and let* $\hat{\boldsymbol{\theta}}$ *be an estimator of the common value* $\boldsymbol{\theta}$ *such that Assumption D holds. Then*

(i) letting $\boldsymbol{\Upsilon}^{(n)} := \left(\sqrt{r_1^{(n)}}\,\mathbf{I}_k \,\vdots\, \sqrt{r_2^{(n)}}\,\mathbf{I}_k\right)'$, $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}}$ *satisfies, under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ *and as* $n \to \infty$,

$$\boldsymbol{\Delta}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{\phi}} - \boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}} = -\boldsymbol{\Gamma}^{\underline{\phi}}_{\boldsymbol{\theta};\underline{g}}\boldsymbol{\Upsilon}^{(n)}\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) + o_{\mathrm{P}}(1),$$

where

$$\boldsymbol{\Gamma}^{\underline{\phi}}_{\boldsymbol{\theta};\underline{g}} := \mathrm{diag}\left(\boldsymbol{\Gamma}^{\phi_{\kappa_1}}_{\boldsymbol{\theta};g_1}, \boldsymbol{\Gamma}^{\phi_{\kappa_2}}_{\boldsymbol{\theta};g_2}\right)$$

with $\boldsymbol{\Gamma}^{\phi_{\kappa_i}}_{\boldsymbol{\theta};g_i} := \kappa_i \mathrm{E}_{g_i}\left[\mathbf{X}_{ij}'\boldsymbol{\theta}\right](\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \quad i = 1, 2;$

ii) for all $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$, $\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{\phi}} = \boldsymbol{\Gamma}^{\underline{\phi}}_{\boldsymbol{\theta};\underline{\phi}}$.

The proof is provided in the Appendix. From these results, we can now deduce the new quadratic form to be used as a $\underline{g}$-valid test statistic. This construction follows the ideas from Hallin and Paindaveine (2008) where a very general theory for pseudo-Gaussian procedures is described. Defining $E_{k,g_i} := \mathrm{E}_{g_i}\left[\mathbf{X}_{ij}'\boldsymbol{\theta}\right]$, $i = 1, 2$, and, for notational simplicity, $D_{k,g_i} := E_{k,g_i}/B_{k,g_i}$, $i = 1, 2$, and $H_{\underline{\phi},\underline{g}} := (r_1^{(n)} D^2_{k,g_1} B_{k,g_1} + r_2^{(n)} D^2_{k,g_2} B_{k,g_2})$, and letting

$$\boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} := (k-1)\begin{pmatrix} \frac{1}{\kappa_1^2}\left(\frac{1}{B_{k,g_1}} - \frac{r_1^{(n)} D^2_{k,g_1}}{H_{\underline{\phi},\underline{g}}}\right)(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') & -\frac{1}{\kappa_1\kappa_2}\frac{\sqrt{r_1^{(n)} r_2^{(n)}} D_{k,g_1} D_{k,g_2}}{H_{\underline{\phi},\underline{g}}}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') \\ -\frac{1}{\kappa_1\kappa_2}\frac{\sqrt{r_1^{(n)} r_2^{(n)}} D_{k,g_1} D_{k,g_2}}{H_{\underline{\phi},\underline{g}}}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') & \frac{1}{\kappa_2^2}\left(\frac{1}{B_{k,g_2}} - \frac{r_2^{(n)} D^2_{k,g_2}}{H_{\underline{\phi},\underline{g}}}\right)(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') \end{pmatrix},$$

the $\underline{g}$-valid test statistic for the two-sample spherical location problem $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ corresponds to the quadratic form

$$Q^{(n)}(\underline{g}) := (\boldsymbol{\Delta}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{\phi}})' \boldsymbol{\Psi}^{\perp}_{\hat{\boldsymbol{\theta}};\underline{\phi},\underline{g}} \boldsymbol{\Delta}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{\phi}}.$$

At first sight, this construction might appear puzzling and the reader might wonder how the preceding results finally lead to this test statistic. It is easy to verify that the random quantity $Q^{(n)}(\underline{g})$ does not depend explicitly on the underlying concentrations κ_1 and κ_2 but still depends on the quantities B_{k,g_i} and E_{k,g_i}, $i = 1, 2$, still hampering the validity of the statistic outside of $\underline{g}$. The last step in our construction thus consists in estimating these quantities. Consistent (via the Law of Large Numbers) estimators for each of them are provided by $\hat{B}_{k,g_i} := 1 - n_i^{-1} \sum_{j=1}^{n_i} (\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})^2$ and $\hat{E}_{k,g_i} := n_i^{-1} \sum_{j=1}^{n_i} (\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})$, $i = 1, 2$. For the sake of readability, we naturally also use the notations $\hat{D}_{k,g_i} := \hat{E}_{k,g_i}/\hat{B}_{k,g_i}$, $i = 1, 2$, and $\hat{H}_{\underline{\phi},\underline{g}} := (r_1^{(n)} \hat{D}^2_{k,g_1} \hat{B}_{k,g_1} + r_2^{(n)} \hat{D}^2_{k,g_2} \hat{B}_{k,g_2})$. Straightforward calculations then show that our pseudo-FvML test statistic for the two-sample spherical location problem is

$$\begin{aligned} Q^{(n)} \;=\; (k-1) \Bigg\{ & \left(\frac{\hat{D}_{k,g_1}}{\hat{E}_{k,g_1}} - \frac{r_1^{(n)} \hat{D}^2_{k,g_1}}{\hat{H}_{\underline{\phi},\underline{g}}} \right) \left(\frac{1}{n_1} \sum_{i,j=1}^{n_1} \mathbf{X}'_{1i} (\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}') \mathbf{X}_{1j} \right) \\ & + \left(\frac{\hat{D}_{k,g_2}}{\hat{E}_{k,g_2}} - \frac{r_2^{(n)} \hat{D}^2_{k,g_2}}{\hat{H}_{\underline{\phi},\underline{g}}} \right) \left(\frac{1}{n_2} \sum_{i,j=1}^{n_2} \mathbf{X}'_{2i} (\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}') \mathbf{X}_{2j} \right) \\ & - 2 \frac{\hat{D}_{k,g_1} \hat{D}_{k,g_2}}{\hat{H}_{\underline{\phi},\underline{g}}} \left(\frac{1}{n} \sum_{i=1}^{n_1} \mathbf{X}'_{1i} (\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}') \sum_{j=1}^{n_2} \mathbf{X}_{2j} \right) \Bigg\}, \end{aligned}$$

which no more depends on $\underline{g}$. The following proposition finally yields the asymptotic properties of this quadratic form under the entire class of rotationally symmetric distributions, showing that the test is well valid under that broad set of distributions.

Proposition 3.3 *Let Assumptions A, B and C hold and let* $\hat{\boldsymbol{\theta}}$ *be an estimator of the common value* $\boldsymbol{\theta}$ *such that Assumption D holds. Then,*

(i) $Q^{(n)}$ *is asymptotically chi-square with* $k-1$ *degrees of freedom under* $\bigcup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} \bigcup_{\underline{g} \in \mathcal{F}^2} \mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$*;*

(ii) $Q^{(n)}$ *is asymptotically non-central chi-square with* $k-1$ *degrees of freedom and non-centrality parameter*

$$l_{(\boldsymbol{\theta},\boldsymbol{\theta}),\mathbf{t};\underline{\phi},\underline{g}} := \mathbf{t}' \boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \mathbf{t}$$

under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$*, where* $\mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2^{(n)}$ *are as in (2.2) and* $\mathbf{t} := (\mathbf{t}'_1, \mathbf{t}'_2)'$ *with* $\mathbf{t}_1 := \lim_{n\to\infty} \mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2 := \lim_{n\to\infty} \mathbf{t}_2^{(n)}$*.*

From Part (i) we deduce that our pseudo-FvML tests for the two-sample spherical location problem, denoted by $\phi^{(n)}$, reject the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ in favor of $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ at asymptotic level α as soon as $Q^{(n)}$ exceeds the α-upper quantile of a chi-square distribution with $k-1$ degrees of freedom. It is easy to verify as in Watson (1983) that it is asymptotically equivalent (the difference is a $o_{\mathrm{P}}(1)$ quantity) to the FvML likelihood ratio test in the FvML case and therefore keeps its optimality properties in that configuration. Thus, our pseudo-FvML tests, although the construction is different, coincide with Watson's proposal. However, our work here is of intrinsic interest since we provide more insight on how to correct the optimal FvML test and, contrarily to Watson (1983), we investigate in detail the impact of the replacement of the unknown quantities by the corresponding estimators.

4. Rank-based tests.

The pseudo-FvML test constructed in the previous section is valid under any pair of (non-necessarily equal) rotationally symmetric distributions and retains the optimality properties of the FvML likelihood ratio test in the FvML case. In this section, we start from any given pair $\underline{f} \in \mathcal{F}^2$ and our objective is to turn the $\underline{f}$-parametric tests into tests which are still valid under any pair of (non-necessarily equal) rotationally symmetric distributions but which are optimal under the pair $\underline{f}$. To obtain such a test, we have recourse here to the second of the aforementioned tools to turn our parametric tests into semi-parametric ones: the invariance principle. This principle advocates that, if the sub-model identified by the null hypothesis is invariant under the action of a group of transformations $\mathcal{G}_T$, one should exclusively use procedures whose outcome does not change along the orbits of that group $\mathcal{G}_T$. This is the case if and only if these procedures are measurable with respect to the maximal invariant associated with $\mathcal{G}_T$. The invariance principle is accompanied by a nice and appealing corollary for our purposes here: provided that the group $\mathcal{G}_T$ is a generating group for $\mathcal{H}_0$, the invariant procedures are distribution-free under the null. In view of all this, our strategy in this section is the following: determine the correct group of transformations $\mathcal{G}_T$, re-express our $\underline{f}$-parametric tests in terms of the corresponding maximal invariant and study the asymptotic properties of the resulting test statistic, after replacement of all the unknown quantities by consistent estimators, as in the previous section.

Invariance with respect to "common rotations" is crucial in this context. More precisely, letting $\mathbf{O} \in \mathcal{SO}_k := \{\mathbf{A} \in \mathbb{R}^{k\times k}, \mathbf{A}'\mathbf{A} = \mathbf{I}_k,\ \det(\mathbf{A}) = 1\}$, the null hypothesis is unquestionably invariant with respect to a transformation of the form

$$g_{\mathbf{O}} : \mathbf{X}_{11}, \ldots, \mathbf{X}_{1n_1}, \mathbf{X}_{21}, \ldots, \mathbf{X}_{2n_2} \mapsto \mathbf{O}\mathbf{X}_{11}, \ldots, \mathbf{O}\mathbf{X}_{1n_1}, \mathbf{O}\mathbf{X}_{21}, \ldots, \mathbf{O}\mathbf{X}_{2n_2}.$$

However, this group is not generating for $\mathcal{H}_0$ as it does not take into account the underlying angular functions $\underline{f}$, which are an infinite-dimensional nuisance under $\mathcal{H}_0$. This group is actually rather generating for $\bigcup_{\boldsymbol{\theta}\in\mathcal{S}^{k-1}} \mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}}$ with fixed $\underline{f}$. Now, denote as in the previous section the common value of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ under the null as $\boldsymbol{\theta}$. Then $\mathbf{X}_{ij} = (\mathbf{X}'_{ij}\boldsymbol{\theta})\boldsymbol{\theta} + \sqrt{1-(\mathbf{X}'_{ij}\boldsymbol{\theta})^2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})$ for all $j = 1, \ldots, n_i$ and $i = 1, 2$. Let

$\mathcal{G}_{\underline{h}}$ ($\underline{h} := (h_1, h_2)$) be the group of transformations of the form

$$\begin{aligned} g_{h_i} : \mathbf{X}_{ij} &\mapsto g_{h_i}(\mathbf{X}_{ij}) \\ &= h_i(\mathbf{X}'_{ij}\boldsymbol{\theta})\boldsymbol{\theta} + \sqrt{1 - (h_i(\mathbf{X}'_{ij}\boldsymbol{\theta}))^2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}), \quad i = 1, 2, \end{aligned}$$

where the $h_i : [-1, 1] \to [-1, 1]$ are monotone continuous nondecreasing functions such that $h_i(1) = 1$ and $h_i(-1) = -1$, $i = 1, 2$. For any pair of (possibly different) transformations $(g_{h_1}, g_{h_2}) \in \mathcal{G}_{\underline{h}}$, it is easy to verify that $\|g_{h_i}(\mathbf{X}_{ij})\| = 1$; thus, g_{h_i} is a monotone transformation from $\mathcal{S}^{k-1}$ to $\mathcal{S}^{k-1}$, $i = 1, 2$. Note furthermore that g_{h_i} does not modify the signs $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})$. It is then quite easy to see that the group of transformations $\mathcal{G}_{\underline{h}}$ is a generating group for $\bigcup_{\underline{f} \in \mathcal{F}^2} \mathrm{P}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{f}}$, which this time corresponds exactly to our null hypothesis $\mathcal{H}_0$, and that the null is invariant under the action of $\mathcal{G}_{\underline{h}}$. A simple exercise reveals that the maximal invariant associated with $\mathcal{G}_{\underline{h}}$ is the vector of signs $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{11}), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{1n_1}), \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{21}), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{2n_2})$ and ranks $R_{11}, \ldots, R_{1n_1}, R_{21}, \ldots, R_{2n_2}$ where R_{ij} denotes the rank of $\mathbf{X}'_{ij}\boldsymbol{\theta}$ among $\mathbf{X}'_{i1}\boldsymbol{\theta}, \ldots, \mathbf{X}'_{in_i}\boldsymbol{\theta}$, $i = 1, 2$. As a consequence, we choose to base our tests in this section on a rank-based version of the central sequence $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{f}}$, namely on

$$\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f} := ((\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\boldsymbol{\theta}; K_{f_1}})', (\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\boldsymbol{\theta}; K_{f_2}})')'$$

with

$$\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\boldsymbol{\theta}; K_{f_i}} = n_i^{-1/2} \sum_{j=1}^{n_i} K_{f_i}\left(\frac{R_{ij}}{n_i + 1}\right) \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}), \quad i = 1, 2,$$

where $\underline{K}_f := (K_{f_1}, K_{f_2})$ is a pair of *score (generating) functions* satisfying

Assumption E. The score functions K_{f_i}, $i = 1, 2$, are continuous functions from $[0, 1]$ to $\mathbb{R}$.

The following result, which is a direct corollary (using again the inner-sample independence and the mutual independence between the two samples) of Proposition 3.1 in Ley *et al.* (2012), characterizes the asymptotic behavior of $\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f}$ under any pair of densities with respective angular functions g_1 and g_2.

Proposition 4.1 *Let Assumptions A, B, C and E hold and consider* $\underline{g} = (g_1, g_2) \in \mathcal{F}^2$. *Then the rank-based central sequence* $\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f}$

(i) is such that $\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f} - \boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f; \underline{g}} = o_{\mathrm{P}}(1)$ *under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{g}}$ *as* $n \to \infty$, *where* ($\tilde{G}_i$ *standing for the common cdf of the* $\mathbf{X}'_{ij}\boldsymbol{\theta}$*'s under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{g}}$, $i = 1, 2$)

$$\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f; \underline{g}} = ((\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}; K_{f_1}; g_1})', (\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}; K_{f_2}; g_2})')'$$

with

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}; K_{f_i}; g_i} := n_i^{-1/2} \sum_{j=1}^{n_i} K_{f_i}\left(\tilde{G}_i(\mathbf{X}'_{ij}\boldsymbol{\theta})\right) \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}), \quad i = 1, 2.$$

In particular, for $\underline{K}_f = (K_{f_1}, K_{f_2})$ *with* $K_{f_i}(u) := \varphi_{f_i}(\tilde{F}_i^{-1}(u))(1 - (\tilde{F}_i^{-1}(u))^2)^{1/2}$, $\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{K}_f}$ *is asymptotically equivalent to the efficient central sequence* $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{f}}$ *under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}, \boldsymbol{\theta}); \underline{f}}$.

(ii) *is asymptotically normal under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ *with mean zero and covariance matrix*

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f} := \mathrm{diag}\left(\frac{\mathcal{J}_k(K_{f_1})}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \frac{\mathcal{J}_k(K_{f_2})}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')\right),$$

where $\mathcal{J}_k(K_{f_i}) := \int_0^1 K^2_{f_i}(u)du$.

(iii) *is asymptotically normal under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$ *(*$\mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2^{(n)}$ *as in (2.2)) with mean* $\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}}\mathbf{t}$ *(*$\mathbf{t} := (\mathbf{t}_1', \mathbf{t}_2')'$ *with* $\mathbf{t}_1 := \lim_{n\to\infty}\mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2 := \lim_{n\to\infty}\mathbf{t}_2^{(n)}$*) and covariance matrix*

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}} := \mathrm{diag}\left(\frac{\mathcal{J}_k(K_{f_1},g_1)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \frac{\mathcal{J}_k(K_{f_2},g_2)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')\right),$$

where $\mathcal{J}_k(K_{f_i}, g_i) := \int_0^1 K_{f_i}(u)K_{g_i}(u)du$ *for* $i = 1, 2$.

(iv) *satisfies, under* $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ *as* $n \to \infty$*, the asymptotic linearity property*

$$\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{K}_f} - \underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{K}_f} = -\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}}\mathbf{t}^{(n)} + o_{\mathrm{P}}(1),$$

for $\mathbf{t}^{(n)} = ((\mathbf{t}_1^{(n)})', (\mathbf{t}_2^{(n)})')'$ *with* $\mathbf{t}_1^{(n)}$ *and* $\mathbf{t}_2^{(n)}$ *as in (2.2).*

Now, as for the pseudo-FvML test, our rank-based procedures are not complete since we still have to estimate the common value $\boldsymbol{\theta}$ of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ under $\mathcal{H}_0$. Therefore, we will assume the existence of an estimator $\hat{\boldsymbol{\theta}}$ satisfying Assumption D; as explained in Section 3, such an estimator is easy to construct. In order to deal with these rank-based test statistics, we however need to strengthen Assumption D into

Assumption D'. Besides root-n consistency under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ for any $\underline{g} \in \mathcal{F}^2$, the estimator $\hat{\boldsymbol{\theta}} \in \mathcal{S}^{k-1}$ is further *locally and asymptotically discrete*, meaning that it only takes a bounded number of distinct values in $\boldsymbol{\theta}$-centered balls with $O(n^{-1/2})$ radius.

This discretization condition is a purely technical requirement (see pages 125 and 188 of Le Cam and Yang 2000 for a discussion), with little practical implications (in fixed-n practice, such discretizations are irrelevant as the radius can be taken arbitrarily large). We will therefore tacitly assume that $\hat{\boldsymbol{\theta}}$ is locally discrete throughout this section. Following Lemma 4.4 in Kreiss (1987), the local discreteness allows to replace in Part (iv) of Proposition 4.1 the non-random perturbations $\boldsymbol{\theta} + n_i^{-1/2}\mathbf{t}_i^{(n)}$, $i = 1, 2$, (of $\boldsymbol{\theta}$) by $\hat{\boldsymbol{\theta}}$ (see also Hallin *et al.* 2011). Based on the asymptotic result of Proposition 4.1 and letting $H_{\underline{K}_f,\underline{g}} := \frac{r_1^{(n)}\mathcal{J}_k^2(K_{f_1},g_1)}{\mathcal{J}_k(K_{f_1})} + \frac{r_2^{(n)}\mathcal{J}_k^2(K_{f_2},g_2)}{\mathcal{J}_k(K_{f_2})}$ and

$$\boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}} := (k-1)\begin{pmatrix} \left(\frac{1}{\mathcal{J}_k(K_{f_1})} - \frac{r_1^{(n)}\mathcal{J}_k^2(K_{f_1},g_1)}{\mathcal{J}_k^2(K_{f_1})H_{\underline{K}_f,\underline{g}}}\right)(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') & -\frac{\sqrt{r_1^{(n)}r_2^{(n)}}\mathcal{J}_k(K_{f_1},g_1)\mathcal{J}_k(K_{f_2},g_2)}{\mathcal{J}_k(K_{f_1})\mathcal{J}_k(K_{f_2})H_{\underline{K}_f,\underline{g}}}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') \\ -\frac{\sqrt{r_1^{(n)}r_2^{(n)}}\mathcal{J}_k(K_{f_1},g_1)\mathcal{J}_k(K_{f_2},g_2)}{\mathcal{J}_k(K_{f_1})\mathcal{J}_k(K_{f_2})H_{\underline{K}_f,\underline{g}}}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') & \left(\frac{1}{\mathcal{J}_k(K_{f_2})} - \frac{r_2^{(n)}\mathcal{J}_k^2(K_{f_2},g_2)}{\mathcal{J}_k^2(K_{f_2})H_{\underline{K}_f,\underline{g}}}\right)(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}') \end{pmatrix},$$

the $\underline{g}$-valid rank-based test statistic (built in a similar way as the pseudo-FvML statistic, see Section 3) we propose for the two-sample spherical location problem $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ corresponds to the quadratic form

$$Q_{\underline{K}_f}(\underline{g})^{(n)} := (\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{K}_f})' \boldsymbol{\Psi}^{\perp}_{\hat{\boldsymbol{\theta}};\underline{K}_f,\underline{g}} \underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}});\underline{K}_f}.$$

The test statistic $Q^{(n)}_{\underline{K}_f}$ still depends on the cross-information quantities $\mathcal{J}_k(K_{f_1}, g_1)$ and $\mathcal{J}_k(K_{f_2}, g_2)$, hence is only valid under fixed $\underline{g}$. Therefore, exactly as for the pseudo-FvML tests, the final step in our construction consists in estimating these quantities consistently. Define, for any $\rho \geq 0$,

$$\tilde{\boldsymbol{\theta}}_i(\rho) := \hat{\boldsymbol{\theta}} + n_i^{-1/2}\rho\,(k-1)(\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}')\,\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\hat{\boldsymbol{\theta}};K_{f_i}}, \quad i = 1, 2. \tag{4.3}$$

Then, letting $\hat{\boldsymbol{\theta}}_i(\rho) := \tilde{\boldsymbol{\theta}}_i(\rho)/\|\tilde{\boldsymbol{\theta}}_i(\rho)\|$, we consider the piecewise continuous quadratic form

$$\rho \mapsto h_i^{(n)}(\rho) := \frac{k-1}{\mathcal{J}(K_{f_i})}(\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\hat{\boldsymbol{\theta}};K_{f_i}})'\,\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\hat{\boldsymbol{\theta}}_i(\rho);K_{f_i}}.$$

As in Ley *et al.* (2012), consistent estimators of $\mathcal{J}_k^{-1}(K_{f_1}, g_1)$ and $\mathcal{J}_k^{-1}(K_{f_2}, g_2)$ (and therefore readily of $\mathcal{J}_k(K_{f_1}, g_1)$ and $\mathcal{J}_k(K_{f_2}, g_2)$) can be obtained by taking respectively $\hat{\rho}_1 := \inf\{\rho > 0 : h_1^{(n)}(\rho) < 0\}$ and $\hat{\rho}_2 := \inf\{\rho > 0 : h_2^{(n)}(\rho) < 0\}$. Denoting by $\hat{\mathcal{J}}_k(K_{f_1}, g_1)$ and $\hat{\mathcal{J}}_k(K_{f_2}, g_2)$ the resulting estimators, $\hat{H}_{\underline{K}_f,\underline{g}} := \frac{r_1^{(n)}\hat{\mathcal{J}}_k^2(K_{f_1},g_1)}{\mathcal{J}_k(K_{f_1})} + \frac{r_2^{(n)}\hat{\mathcal{J}}_k^2(K_{f_2},g_2)}{\mathcal{J}_k(K_{f_2})}$ and letting $\mathbf{U}_{ij} := K_{f_i}\left(\frac{\hat{R}_{ij}}{n_i+1}\right)\mathbf{S}_{\hat{\boldsymbol{\theta}}}(\mathbf{X}_{ij})$, $i = 1, 2$, ($\hat{R}_{ij}$ naturally stands for the rank of $\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}}$ among $\mathbf{X}'_{i1}\hat{\boldsymbol{\theta}}, \ldots, \mathbf{X}'_{in_i}\hat{\boldsymbol{\theta}}$), the proposed rank test $\phi^{(n)}_{\underline{K}_f}$ rejects the null hypothesis of homogeneity of the locations when

$$\begin{aligned} Q^{(n)}_{\underline{K}_f} \;:=\; (k-1)\Bigg\{ & \left(\frac{1}{\mathcal{J}_k(K_{f_1})} - \frac{r_1^{(n)}\hat{\mathcal{J}}_k^2(K_{f_1}, g_1)}{\mathcal{J}_k^2(K_{f_1})\hat{H}_{\underline{K}_f,\underline{g}}}\right)\left(\frac{1}{n_1}\sum_{i,j=1}^{n_1}\mathbf{U}'_{1i}(\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}')\mathbf{U}_{1j}\right) \\ & + \left(\frac{1}{\mathcal{J}_k(K_{f_2})} - \frac{r_2^{(n)}\hat{\mathcal{J}}_k^2(K_{f_2}, g_2)}{\mathcal{J}_k^2(K_{f_2})\hat{H}_{\underline{K}_f,\underline{g}}}\right)\left(\frac{1}{n_2}\sum_{i,j=1}^{n_2}\mathbf{U}'_{2i}(\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}')\mathbf{U}_{2j}\right) \\ & - 2\frac{\hat{\mathcal{J}}_k(K_{f_1}, g_1)\hat{\mathcal{J}}_k(K_{f_2}, g_2)}{\mathcal{J}_k(K_{f_1})\mathcal{J}_k(K_{f_2})\hat{H}_{\underline{K}_f,\underline{g}}}\left(\frac{1}{n}\sum_{i=1}^{n_1}\mathbf{U}'_{1i}(\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}')\sum_{j=1}^{n_2}\mathbf{U}_{2j}\right)\Bigg\} \end{aligned}$$

exceeds the α-upper quantile of the chi-square distribution with $k-1$ degrees of freedom. This asymptotic behavior under the null as well as the asymptotic distribution of $Q^{(n)}_{\underline{K}_f}$ under a sequence of contiguous alternatives are summarized in the following proposition.

Proposition 4.2 *Let Assumptions A, B, C and E hold and let $\hat{\boldsymbol{\theta}}$ be an estimator of the common value $\boldsymbol{\theta}$ such that Assumption D' holds. Then*

(i) $Q^{(n)}_{\underline{K}_f}$ is asymptotically chi-square with $k-1$ degrees of freedom under $\bigcup_{\boldsymbol{\theta}\in\mathcal{S}^{k-1}}\bigcup_{\underline{g}\in\mathcal{F}^2}\{\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}\}$;

(ii) $Q^{(n)}_{\underline{K}_f}$ is asymptotically non-central chi-square, still with $k-1$ degrees of freedom, but with non-centrality parameter

$$l_{(\boldsymbol{\theta},\boldsymbol{\theta}),\mathbf{t};\underline{K}_f,\underline{g}} := \mathbf{t}'\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}}\boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}}\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{K}_f,\underline{g}}\mathbf{t}$$

under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$, where $\mathbf{t}_1^{(n)}$ and $\mathbf{t}_2^{(n)}$ are as in (2.2) and $\mathbf{t} := (\mathbf{t}'_1, \mathbf{t}'_2)'$ with $\mathbf{t}_1 := \lim_{n\to\infty}\mathbf{t}_1^{(n)}$ and $\mathbf{t}_2 := \lim_{n\to\infty}\mathbf{t}_2^{(n)}$.

Thanks to Proposition 4.1, the proof of this result follows along the same lines as that of Proposition 3.3 and is therefore omitted. Exactly as the pseudo-FvML tests $\phi^{(n)}$, our rank-based tests $\phi^{(n)}_{\underline{K}_f}$ are valid under any (non-necessarily equal) pair of rotationally symmetric densities. Furthermore, as shown in Part (i) of Proposition 4.1, for $\underline{K}_f = (K_{f_1}, K_{f_2})$ with $K_{f_i}(u) = \varphi_{f_i}(\tilde{F}_i^{-1}(u))(1-(\tilde{F}_i^{-1}(u))^2)^{1/2}$, $i = 1, 2$, the rank-based central sequence $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{K}_f}$ is asymptotically equivalent to the parametric central sequence $\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}}$ under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{f}}$. Therefore, the test $\phi^{(n)}_{\underline{K}_f}$ based on the central sequence $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{K}_f}$ keeps the optimality properties of the $\underline{f}$-parametric test for any $\underline{f} \in \mathcal{F}^2$. Thus, while the pseudo-FvML tests are logically FvML-based and only enjoy optimality under FvML densities, one can construct $\underline{f}$-optimal rank-based tests on basis of any pair $\underline{f} \in \mathcal{F}^2$. This, in particular, provides practitioners with much more flexibility than was previously available.

We conclude this section by comparing the optimal pseudo-FvML test $\phi^{(n)}$ with optimal rank-based tests $\phi^{(n)}_{\underline{K}_f}$ for several choices of $\underline{f} \in \mathcal{F}^2$ by means of Pitman's asymptotic relative efficiency (ARE). Letting $\mathrm{ARE}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}(\phi^{(n)}_1, \phi^{(n)}_2)$ denote the ARE of a test $\phi^{(n)}_1$ with respect to another test $\phi^{(n)}_2$ under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$, we have that

$$\mathrm{ARE}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}(\phi^{(n)}_{\underline{K}_f}, \phi^{(n)}) = l_{(\boldsymbol{\theta},\boldsymbol{\theta}),\mathbf{t};\underline{K}_f,\underline{g}} / l_{(\boldsymbol{\theta},\boldsymbol{\theta}),\mathbf{t};\underline{\phi},\underline{g}}.$$

In the homogeneous case $\underline{g} = (g_1, g_1)$ (the angular density is the same for both samples) and if the same score function—namely, K_{f_1}—is used for the two rankings (the test is therefore denoted by $\phi^{(n)}_{K_{f_1}}$), the ratio in (4) simplifies into

$$\mathrm{ARE}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}(\phi^{(n)}_{K_{f_1}} / \phi^{(n)}) = \frac{\mathcal{J}_k^2(K_{f_1}, g_1)}{\mathcal{J}_k(K_{f_1}) D^2_{k,g_1} B_{k,g_1}}. \tag{4.4}$$

Numerical values of the AREs in (4.4) are reported in Table 1 in the three-dimensional case under various angular densities and various choices of the score function K_{f_1}. More precisely, we consider the spherical linear, logarithmic and logistic distributions with respective angular functions

$$f_{\mathrm{lin(a)}}(t) := t + a, \quad f_{\mathrm{log(a)}}(t) := \log(t + a) \quad \text{and}$$

$$f_{\mathrm{logis}(a,b)}(t) := \frac{a\,\exp(-b\,\arccos(t))}{(1 + a\,\exp(-b\,\arccos(t)))^2}.$$

The constants a and b are chosen so that all the above functions are true angular functions satisfying Assumption A. The score functions associated with these angular functions are denoted by $K_{\mathrm{lin(a)}}$ for $f_{\mathrm{lin(a)}}$, $K_{\mathrm{log(a)}}$ for $f_{\mathrm{log(a)}}$ and $K_{\mathrm{logis(a,b)}}$ for $f_{\mathrm{logis(a,b)}}$. For the FvML distribution with concentration κ, the score function will be denoted by K_{ϕ_κ}.

Inspection of Table 1 confirms the theoretical results. As expected, the pseudo-FvML test $\phi^{(n)}$ dominates the rank-based tests under FvML densities, whereas rank-based tests mostly outperform the pseudo-FvML test under other densities, especially so when they are based on the score function associated with the underlying density (in which case the rank-based tests are optimal).

	$\mathrm{ARE}(\phi^{(n)}_{K_{f_1}}/\phi^{(n)})$						
Underlying density	$\phi^{(n)}_{K_{\phi_2}}$	$\phi^{(n)}_{K_{\phi_6}}$	$\phi^{(n)}_{K_{\mathrm{lin}(2)}}$	$\phi^{(n)}_{K_{\mathrm{lin}(4)}}$	$\phi^{(n)}_{K_{\log(2.5)}}$	$\phi^{(n)}_{K_{\mathrm{logis}(1,1)}}$	$\phi^{(n)}_{K_{\mathrm{logis}(2,1)}}$
FvML(1)	0.9744	0.8787	0.9813	0.9979	0.9027	0.9321	0.7364
FvML(2)	1	0.9556	0.9978	0.9586	0.9749	0.9823	0.8480
FvML(6)	0.9555	1	0.9381	0.8517	0.9768	0.9911	0.9280
Lin(2)	1.0539	0.9909	1.0562	1.0215	1.0212	1.0247	0.8796
Lin(4)	0.9709	0.8627	0.9795	1.0128	0.8856	0.9231	0.7097
Log(2.5)	1.1610	1.1633	1.1514	1.0413	1.1908	1.1625	1.0951
Log(4)	1.0182	0.9216	1.0261	1.0347	0.9503	0.9741	0.7851
Logis(1,1)	1.0768	1.0865	1.0635	0.9991	1.0701	1.0962	0.9778
Logis(2,1)	1.3182	1.4426	1.2946	1.0893	1.4294	1.3865	1.5544

Table 1: Asymptotic relative efficiencies of (homogeneous) rank-based tests $\phi^{(n)}_{K_{f_1}}$ with respect to the pseudo-FvML test $\phi^{(n)}$ under various three-dimensional rotationally symmetric densities.

5. Simulation results

In this section, we perform a Monte Carlo study to compare the small-sample behavior of the pseudo-FvML test $\phi^{(n)}$ and various rank-based tests $\phi^{(n)}_{\underline{K}_f}$. For this purpose, we generated $M = 2,500$ replications of four pairs of mutually independent samples (with respective sizes $n_1 = 100$ and $n_2 = 150$) of three-dimensional rotationally symmetric random vectors

$$\boldsymbol{\varepsilon}_{\ell;ij_i}, \quad \ell = 1,2,3,4, \quad j_i = 1,\ldots,n_i, \quad i = 1,2,$$

with FvML densities and linear densities: the $\boldsymbol{\varepsilon}_{1;1j_1}$'s have a FvML(15) distribution and the $\boldsymbol{\varepsilon}_{1;2j_2}$'s have a FvML(2) distribution; the $\boldsymbol{\varepsilon}_{2;1j_1}$'s have a Lin(2) distribution and the $\boldsymbol{\varepsilon}_{2;2j_2}$'s have a Lin(1.1) distribution; the $\boldsymbol{\varepsilon}_{3;1j_1}$'s have a FvML(15) distribution and the $\boldsymbol{\varepsilon}_{3;2j_2}$'s have a Lin(1.1) distribution and finally the $\boldsymbol{\varepsilon}_{4;1j_1}$'s have a Lin(2) distribution and the $\boldsymbol{\varepsilon}_{4;2j_2}$'s have a FvML(2) distribution.

The rotationally symmetric vectors $\boldsymbol{\varepsilon}_{\ell;ij_i}$'s have all been generated with a common spherical location $\boldsymbol{\theta}_0 = (\sqrt{3}/2, 1/2, 0)'$. Then, each replication of the $\boldsymbol{\varepsilon}_{\ell;ij_i}$'s was transformed into

$$\begin{cases} \mathbf{X}_{\ell;1j_1} = \boldsymbol{\varepsilon}_{\ell;1j_1}, \quad \ell = 1,2,3,4, \quad j_1 = 1,\ldots,n_1 \\ \mathbf{X}_{\ell;2j_2;\xi} = \mathbf{O}_\xi \boldsymbol{\varepsilon}_{\ell;2j_2}, \quad \ell = 1,2,3,4, \quad j_2 = 1,\ldots,n_2, \quad \xi = 0,1,2,3, \end{cases}$$

where

$$\mathbf{O}_\xi = \begin{pmatrix} \cos(\pi\xi/16) & -\sin(\pi\xi/16) & 0 \\ \sin(\pi\xi/16) & \cos(\pi\xi/16) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the spherical locations of the $\mathbf{X}_{\ell;1j_1}$'s and the $\mathbf{X}_{\ell;2j_2;0}$'s coincide while the spherical location of the $\mathbf{X}_{\ell;2j_2;\xi}$'s, $\xi = 1,2,3$, is different from the spherical location of the $\mathbf{X}_{\ell;1j_1}$'s characterizing alternatives to the null hypothesis of common spherical locations. Rejection frequencies based on the asymptotic chi-square critical values at nominal level 5% are reported in Table 2 below. The inspection of the latter reveals expected results:

(i) The pseudo-FvML test and all the rank-based tests are valid under heterogeneous densities. They reach the 5% nominal level constraint under any considered pair of densities.

(ii) The comparison of the empirical powers reveals that when based on scores associated with the underlying distributions, the rank-based test performs nicely. The pseudo-FvML test is clearly optimal in the FvML case.

Appendix A.

Proof of Proposition 3.1. From Watson (1983) (and the beginning of Section 2) we know that, under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$, the sign vectors $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})$ are independent of the scalar products $\mathbf{X}'_{ij}\boldsymbol{\theta}$, $\mathrm{E}_{g_i}[\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})] = 0$ and that

$$\mathrm{E}_{g_i}\left[\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})\mathbf{S}'_{\boldsymbol{\theta}}(\mathbf{X}_{ij})\right] = \frac{1}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')$$

for $i = 1, 2$ and for all $j = 1, \ldots, n_i$. These results readily allow to obtain Part (i) by applying the multivariate central limit theorem, while Part (ii) follows from the ULAN structure of the model in Proposition 2.1 and Le Cam's third Lemma. □

Proof of Proposition 3.2. We start by proving Part (i). First note that easy computations yield (for $i = 1, 2$)

$$\begin{aligned}
\boldsymbol{\Delta}^{(n)}_{\hat{\boldsymbol{\theta}};\phi_{\kappa_i}} &= \kappa_i n_i^{-1/2}\sum_{j=1}^{n_i}\left[\mathbf{X}_{ij} - (\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}\right] \\
&= \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta};\phi_{\kappa_i}} - \kappa_i n_i^{-1/2}\sum_{j=1}^{n_i}\left[(\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} - (\mathbf{X}'_{ij}\boldsymbol{\theta})\boldsymbol{\theta}\right] \\
&= \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta};\phi_{\kappa_i}} - \mathbf{V}_i^{(n)} - \mathbf{W}_i^{(n)},
\end{aligned}$$

where $\mathbf{V}_i^{(n)} := \kappa_i n_i^{-1}\sum_{j=1}^{n_i}\left[\mathbf{X}'_{ij}\boldsymbol{\theta}\right] n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ and $\mathbf{W}_i^{(n)} := \hat{\boldsymbol{\theta}}\,\kappa_i n_i^{-1}(\sum_{j=1}^{n_i}\mathbf{X}'_{ij})n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Now, combining the delta method (recall that $\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'$ is the Jacobian matrix of the mapping $h : \mathbb{R}^k \to \mathcal{S}^{k-1} : \mathbf{x} \mapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}$ evaluated at $\boldsymbol{\theta}$), the Law of Large Numbers and Slutsky's Lemma, we obtain that

$$\begin{aligned}
\mathbf{V}_i^{(n)} &= \left(\kappa_i n_i^{-1}\sum_{j=1}^{n_i}\mathbf{X}'_{ij}\boldsymbol{\theta}\right) n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \\
&= \kappa_i \mathrm{E}_{g_i}[\mathbf{X}'_{ij}\boldsymbol{\theta}]\,(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_{\mathrm{P}}(1) \\
&= \boldsymbol{\Gamma}^{\phi_{\kappa_i}}_{\boldsymbol{\theta};g_i} n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_{\mathrm{P}}(1)
\end{aligned}$$

		ξ			
Test	True densities	0	1	2	3
$\phi^{(n)}$	(ϕ_{15}, ϕ_2)	.0592	.2684	.8052	.9888
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\phi_2})}$		.0696	.2952	.8276	.9900
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\text{Lin}(1.1)})}$		.0536	.2316	.7660	.9756
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\phi_2})}$		.0656	.2952	.8160	.9894
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\text{Lin}(1.1)})}$		.0544	.2308	.7716	.9772
$\phi^{(n)}$	$(\text{Lin}(2), \text{Lin}(1.1))$	.0480	.0596	.0792	.1312
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\phi_2})}$		.0472	.0568	.0948	.1340
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\text{Lin}(1.1)})}$		.0464	.0604	.0892	.1424
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\phi_2})}$		.0520	.0588	.0920	.1440
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\text{Lin}(1.1)})}$		.0480	.0580	.0856	.1340
$\phi^{(n)}$	$(\text{Lin}(2), \phi_2)$	.0508	.0684	.1044	.1512
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\phi_2})}$		.0540	.0648	.1012	.1532
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\text{Lin}(1.1)})}$		.0512	.0664	.1084	.1608
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\phi_2})}$		.0508	.0656	.1072	.1620
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\text{Lin}(1.1)})}$		.0496	.0628	.1004	.1516
$\phi^{(n)}$	$(\phi_{15}, \text{Lin}(1.1))$	.0468	.1008	.2908	.5760
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\phi_2})}$		.0628	.1288	.3612	.6788
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\text{Lin}(1.1)})}$		.0512	.1156	.3636	.6892
$\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\phi_2})}$		.0616	.1220	.3620	.6768
$\phi^{(n)}_{(K_{\phi_{15}}, K_{\text{Lin}(1.1)})}$		.0504	.1180	.3660	.6916

Table 2: Rejection frequencies (out of $M = 2,500$ replications), under the null and under increasingly distant alternatives, of the pseudo-FvML test $\phi^{(n)}$ and various rank-based tests $\phi^{(n)}_{(K_{\phi_{15}}, K_{\phi_2})}$ (based on FvML(15) and FvML(2) scores), $\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\text{Lin}(1.1)})}$ (based on Lin(2) and Lin(1.1) scores), $\phi^{(n)}_{(K_{\text{Lin}(2)}, K_{\phi_2})}$ (based on Lin(2) and FvML(2) scores) and $\phi^{(n)}_{(K_{\phi_{15}}, K_{\text{Lin}(1.1)})}$ (based on FvML(15) and Lin(1.1) scores). Sample sizes are $n_1 = 100$ and $n_2 = 150$.

under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ as $n \to \infty$. Thus, the announced result follows as soon as we have shown that $\mathbf{W}_i^{(n)}$ is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ as $n \to \infty$. Using the same arguments as for $\mathbf{V}_i^{(n)}$, we have under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ and for $n \to \infty$ that

$$\begin{aligned}
\mathbf{W}_i^{(n)} &= \hat{\boldsymbol{\theta}} \left(\kappa_i n_i^{-1} \sum_{j=1}^{n_i} \mathbf{X}'_{ij} \right) n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \\
&= \hat{\boldsymbol{\theta}} \left(\kappa_i n_i^{-1} \sum_{j=1}^{n_i} (\mathbf{X}'_{ij}) \left(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'\right) \right) n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_{\mathrm{P}}(1) \\
&= \hat{\boldsymbol{\theta}}\, \kappa_i \mathrm{E}_{g_i} \left[\sqrt{1 - (\mathbf{X}'_{ij}\boldsymbol{\theta})^2} (\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}))' \right] n_i^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_{\mathrm{P}}(1),
\end{aligned}$$

which is $o_{\mathrm{P}}(1)$ from the boundedness of $\hat{\boldsymbol{\theta}}$ and since from Watson (1983) (see the proof of Proposition 3.1 for more details) we know that

$$\mathrm{E}_{g_i} \left[\sqrt{1 - (\mathbf{X}'_{ij}\boldsymbol{\theta})^2} (\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}))' \right] = \mathrm{E}_{g_i} \left[\sqrt{1 - (\mathbf{X}'_{ij}\boldsymbol{\theta})^2} \right] \mathrm{E}_{g_i} \left[(\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}))' \right] = \mathbf{0}'.$$

This concludes Part (i) of the proposition. Regarding Part (ii), let $\mathbf{X}$ be a random vector distributed according to an FvML distribution with concentration κ. Then, writing c for the normalization constant, a simple integration by parts yields

$$\begin{aligned}
C_{k,\phi_\kappa} = \kappa\, \mathrm{E}_{\phi_\kappa}[1 - (\mathbf{X}'\boldsymbol{\theta})^2] &= \kappa\, c \int_{-1}^{1} (1 - u^2) \exp(\kappa u)(1 - u^2)^{(k-3)/2}\, du \\
&= \kappa\, c \int_{-1}^{1} \exp(\kappa u)(1 - u^2)^{(k-1)/2}\, du \\
&= c(k-1) \int_{-1}^{1} u \exp(\kappa u)(1 - u^2)^{(k-3)/2}\, du \\
&= (k-1)\, \mathrm{E}_{\phi_\kappa}[\mathbf{X}'\boldsymbol{\theta}].
\end{aligned}$$

The claim thus holds. □

Proof of Proposition 3.3. We start the proof by showing that the replacement of $\boldsymbol{\theta}$ with $\hat{\boldsymbol{\theta}}$ as well as the distinct estimators have no asymptotic cost on $Q^{(n)}$. The consistency of $\hat{D}_{k,g_i}$, $\hat{E}_{k,g_i}$, $i = 1, 2$, and $\hat{H}_{\underline{\phi},\underline{g}}$ together with the $\sqrt{n}$-consistency of $\hat{\boldsymbol{\theta}}$ entail that, using Part (i) of Proposition 3.2,

$$Q^{(n)} = \left(\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}} - \boldsymbol{\Gamma}^{\phi}_{\boldsymbol{\theta};\underline{g}} \boldsymbol{\Upsilon}^{(n)} \sqrt{n} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \right) \right)' \boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \left(\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}} - \boldsymbol{\Gamma}^{\phi}_{\boldsymbol{\theta};\underline{g}} \boldsymbol{\Upsilon}^{(n)} \sqrt{n} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \right) \right) + o_{\mathrm{P}}(1)$$

under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$ as $n \to \infty$. Now, standard algebra yields that $\boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \boldsymbol{\Gamma}^{\phi}_{\boldsymbol{\theta};\underline{g}} \boldsymbol{\Upsilon}^{(n)} = (\boldsymbol{\Gamma}^{\phi}_{\boldsymbol{\theta};\underline{g}} \boldsymbol{\Upsilon}^{(n)})' \boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} = \mathbf{0}$, so that

$$\begin{aligned}
Q^{(n)} &= \left(\boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}} \right)' \boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}} \boldsymbol{\Delta}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{\phi}} + o_{\mathrm{P}}(1) \\
&=: Q^{(n)}(\boldsymbol{\theta}) + o_{\mathrm{P}}(1).
\end{aligned}$$

Both results from Proposition 3.1 entail that since $\boldsymbol{\Gamma}_{\boldsymbol{\theta};\underline{g}} \boldsymbol{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}}$ is idempotent with trace $(k-1)$, $Q^{(n)}(\boldsymbol{\theta})$ (and therefore $Q^{(n)}$) is asymptotically chi-square with $(k-1)$ degrees of freedom under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta},\boldsymbol{\theta});\underline{g}}$, and asymptotically non-central chi-square, still with $(k-1)$ degrees of freedom, and with non-centrality parameter $\mathbf{t}'\mathbf{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}}\mathbf{\Psi}^{\perp}_{\boldsymbol{\theta};\underline{\phi},\underline{g}}\mathbf{\Gamma}_{\boldsymbol{\theta};\underline{\phi},\underline{g}}\mathbf{t}$ under $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}+n_1^{-1/2}\mathbf{t}_1^{(n)},\boldsymbol{\theta}+n_2^{-1/2}\mathbf{t}_2^{(n)});\underline{g}}$. □

Acknowledgements

The research of Christophe Ley is supported by a Mandat de Chargé de Recherche from the Fonds National de la Recherche Scientifique, Communauté française de Belgique.

Arnold, K. J. (1941). *On spherical probability distributions.* PhD thesis, Massachusetts Institute of Technology.

Azzalini, A. and Genton, M. G. (2007). On Gauss's characterization of the normal distribution. *Bernoulli* **13**, 169–174.

Beran, R. and Fisher, N. I. (1998). Nonparametric comparison of mean directions or mean axes. *Ann. Statist.* **26**, 472–493.

Bingham, M. S. and Mardia, K. V. (1975). Characterizations and applications. In *S. Kotz, G. P. Patil and J. K. Ord, eds, Statistical Distributions for Scientific Work, volume 3, Reidel, Dordrecht and Boston*, 387–398.

Breitenberger, E. (1963). Analogues of the normal distribution on the circle and the sphere. *Biometrika* **50**, 81–88.

Chang, T. (2004). Spatial statistics. *Statist. Sci.* **19**, 624–635.

Duerinckx, M. and Ley, C. (2012). Maximum likelihood characterization of rotationally symmetric distributions. *Sankhyā Ser. A, to appear.*

Fisher, R. A. (1953). Dispersion on a sphere. *Proceedings of the Royal Society of London, Ser. A* **217**, 295–305.

Fisher, N. I., Lewis, T., and Embleton, B. J. J. (1987). *Statistical Analysis of Spherical Data.* Cambridge University Press, UK.

Gauss, C. F. (1809). *Theoria motus corporum coelestium in sectionibus conicis solem ambientium.* Hamburg: Perthes et Besser. English translation by C.H. Davis, reprinted by Dover, New York (1963).

Hallin, M. and Paindaveine, D. (2008). A general method for constructing pseudo-Gaussian tests. *J. Japan Statist. Soc.* **38**, 27–39.

Hallin, M., Paindaveine, D. and Verdebout, T. (2010). Optimal rank-based testing for principal components. *Ann. Stat.* **38**, 3245–3299.

Hallin, M., Swan, Y., Verdebout, T. and Veredas, D. (2011). One-step R-estimation in linear models with stable errors. *Ann. J. Econometrics, to appear.*

Kreiss, J. P. (1987). On adaptive estimation in stationary ARMA processes. *Ann. Stat.* **15**, 112–133.

Langevin, P. (1905). *J. Phys.* **4**, 678–693; *Ann. Chim. Phys.* **5**, 70–127.

Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory.* Springer-Verlag, New York.

Le Cam, L. and Yang, G. L. (2000). *Asymptotics in Statistics*, 2nd edition. Springer-Verlag, New York.

Leong, P. and Carlile, S. (1998). Methods for spherical data analysis and visualization. *J. Neurosci. Meth.* **80**, 191–200.

Ley, C., Swan, Y., Thiam, B. and Verdebout, T. (2012). Optimal R-estimation of a spherical location. *Statist. Sinica, to appear.*

Mardia, K. V. and Jupp, P. E. (2000). *Directional Statistics.* Wiley, New York.

McFadden, P. L. and Jones, D. L. (1981). The fold test in palaeomagnetism. *Geophys. J. R. Astron. Soc.* **67**, 53–58.

Muirhead, R. J. and Waternaux, C. M. (1980). Asymptotic distributions in canonical correlation analysis and other multivariate procedures for nonnormal populations. *Biometrika* **67**, 31–43.

Saw, J. G. (1978). A family of distributions on the m-sphere and some hypothesis tests. *Biometrika* **65**, 69–73.

Schaeben, H. (1992). "Normal" orientation distributions. *Textures Microstruct.* **19**, 197–202.

Tsai, M. T. (2009). Asymptotically efficient two-sample rank tests for modal directions on spheres. *J. Multivariate Anal.* **100**, 445–458.

von Mises, R. (1918). *Physikalische Zeitschrift* **19**, 490–500.

Watson, G. S. (1983). *Statistics on Spheres.* Wiley, New York.

Wellner, J. A. (1979). Permutation tests for directional data. *Ann. Statist.* **7**, 929–943.

Documents de travail récents

- Jean-Philippe Garnier: "Social status, a new source of fluctuations?" [2012-15]
- Jean-Philippe Garnier: "Sunspots, cycles and adjustment costs in the two-sectors model"[2012-14]
- François Langot, Lise Patureau and Thepthida Sopraseuth : "Optimal Fiscal Devaluation" [2012-13]
- Marc Germain: "Equilibres et effondrement dans le cadre d'un cycle naturel" [2012-12]
- Marc Hallin, Davy Paindaveine and Thomas Verdebout: "Optimal Rank-based Tests for Common Principal Components" [2012-11]
- Carlotta Balestra, Thierry Bréchet and Stéphane Lambrecht : "Property rights with biological spillovers: when Hardin meets Meade " [2012-10]
- Kirill Borissov, Thierry Bréchet, Stéphane Lambrecht : "Environmental Maintenance in a dynamic model with Heterogenous Agents" [2012-9]
- Nicolas Fleury et Fabrice Gilles : "Mobilités intergénérationnelles de capital humain et restructurations industrielles. Une évaluation pour le cas de la France, 1946-1999" [2012-8]
- Claire Naiditch, Agnes Tomini and Christian Ben Lakhdar "Remittances and incentive to migrate: An epidemic approach of migration" [2012-7]
- Nicolas Berman, Antoine Berthou and Jérôme Héricourt: "Export dynamics and sales at home" [2012-6]
- Muhammad Azmat Hayat, Etienne Farvaque: "Public Attitudes towards Central Bank Independence:Lessons From the Foundation of the ECB" [2012-5]
- Amandine Ghintran, Enrique Gonzales–Arangüena and Conrado Manuel : "A Probabilistic position value" [2011-4]
- Sophie Dabo-Niang, Anne-Françoise Yao : "Kernel spatial density estimation in infinite dimension" [2011-3]
- Xavier Chojnicki, Lionel Ragot: "Impacts of Immigration on Aging Welfare-State - An Applied General Equilibrium Model for France" [2011-2]