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The dynamics of income inequality in a growth model with human capital and occupational choice*

Kirill Borissov[†]and Stéphane Lambrecht[‡]

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Abstract

We model a successive-generation economy in which parents, motivated by family altruism, decide to finance or not their offspring's human capital accumulation on the basis of their altruistic motive, their own income and the equilibrium ratio between skilled labor and unskilled labor wages. The question we ask is how the growth process shapes the wage inequality and the split of the population in two classes. We study the transitional dynamics of human capital accumulation and of income inequality. First, we prove the existence of equilibrium paths. Then we show that there exists a continuum of steady-state equilibria. We prove the convergence of each equilibrium path to one of the steady-state equilibrium and describe the evolution of income inequality between skilled and unskilled workers and income inequality among the skilled workers. The former inequality is persistent in the long run while the latter is not. We also look at the relationship between inequality and output on the set of steady states and find that this relationship is ambiguous. Finally, we develop an endogenous-growth version of the model. In this version of the model the relationship between inequality and the rate of growth is also ambiguous.

Keywords: Education; occupational choice; growth; Inequality
JEL classification: O41; D31; D91; D64; I24; J24

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1 Introduction

The understanding of the role of human capital in economic activity has been decisively spurred by the work of G. Becker (1964). Human capital is a major feature of economic relations inside the family and, as such, is one of the key concepts for understanding of the individual decision-making over the life-cycle and the functioning of labor markets. On their side, macroeconomists and especially economic growth theorists investigated the role of human capital in determining the growth rate of economies in the short and the long run[1].

Modern growth theory has long been relying on the fiction of the representative agent. However, societies are patently not homogeneous, whether in incomes, wealth, or many other dimensions. In a sense, the question of inequality and its link with the growth process is an old one. The classical argument is that inequality is good for growth because the wealthy are more patient and accumulate more assets than the poor. Over the last 20 or 30 years, the nexus between inequality and growth has attracted a great deal of interest. We can distinguish two types of questions about this nexus. The first one is: "How does inequality affect growth?", namely do unequal economies perform better than those more equal? The policy implications of this first type of question are relevant for policies aiming at redistributing income and wealth among households. The role played by capital market imperfection in discouraging human capital accumulation has been stressed by several important contributions (see, e.g., Loury (1981), Galor and Zeira (1993))[2].

The second type of question about the inequality-growth nexus is then the following: "How does the growth process affect in turn inequality?", namely what is the feedback of growth into the evolution of inequality across time. This second question is relevant also for the first one. Indeed, as Aghion et al. (1999) argue, if redistribution creates a virtuous circle by alleviating credit constraints to human capital accumulation, these policy efforts might be vain if growth in turn worsen inequalities. A virtuous circle would be more or less offset by a vicious circle.

This paper deals the inequality-growth relationship by introducing occupational heterogeneity. In our model economy workers may occupy positions of different skill levels and get different wages. As a consequence, educational decisions determine not only the individual stock of human capital but also influences the choice of occupation and the wages structure.

Ray (2006) examines equilibrium paths of an economy in which skilled and unskilled labor are necessary to produce. Each generation decides whether to finance the offspring's acquisition of human capital out of a dynastic (Barro, 1974) altruistic motive to finance educational expenses. Since both skilled and unskilled labor are necessary for production, equilibrium wages adjust to insure that each category of labor has positive supply, i.e. that one share of the population occupies unskilled positions and the other skilled positions. As a result, inequality inside each generation must emerge. Ray (2006) shows

[1]To mention but one major contribution, see, *e.g.*, Lucas (1988).

[2]Other arguments have been put forward to emphasize the negative relationship between inequality and growth: political economy arguments (Alesina and Rodrik (1994), Bertola (1993), Persson and Tabellini (1994)), or social conflict arguments (Alesina and Perotti (1996) and Borissov and Lambrecht (2009)).

that this intragenerational inequality is persistent in the long run. Moreover there exists a continuum of steady state equilibria.

In a recent contribution, Mookherjee and Ray (forthcoming) extend and modify Ray's (2006) model. They provide a small-open economy model with physical capital, a continuum of occupations, training costs and a mix of utility-based and wealth-based motivation for bequest. They derive conditions under which the steady state exhibits inequality. These conditions rely on the share of occupations with high training costs[3] being non degenerate.

Our article is close to Ray's (2006) and Mookherjee and Ray's (forthcoming) contribution. Indeed it looks at the evolution and the persistence of inequality when agents are altruists and face occupational heterogeneity. Its main contribution is the analysis of the endogenous evolution of inequality between skilled and unskilled workers and among skilled workers.

This is worth a further comment. Models with skilled and unskilled workers can be divided in two types. The first type of approach is based on the hypothesis of a fixed education cost. The skilled workers are those who decide to pay for this cost. In equilibrium there only are two different levels of human capital and the production process uses these two types production factors (Galor and Zeira (1993)). The second approach consists in assuming that education expenditure is a continuous variable and that individuals decide how much human capital to invest. All workers are then perfect substitutes in the production process but differ by the number of efficiency units of labor they own (Glomm and Ravikumar (1992)).

Our model differs from these two approaches. We assume two types of labor, skilled and unskilled, but skilled workers are not all identical : they can differ by the level of their human capital. Moreover there is no fixed education cost. Individuals decide to invest in their offspring's education by taking into account their own human capital and the ratio of total wages their offspring would get in skilled and unskilled occupations. Since this wage ratio is endogenous, this is the channel through which the growth process influences human capital investment and inequality on the transition and in the long run. In the literature models combining educational investment and different types of occupations (e.g. Ray (2006) and Mookherjee and Ray (forthcoming)) typically do not rely on a *dynamic process of human capital accumulation*. In these papers education is a *flow*, not a *stock*. Our approach with human capital as a stock allows us to model the influence of parents' own human capital stock through the education function[4]. Moreover within such a framework we are able to study the dynamics of income inequality among the skilled workers.

We assume that individuals care about their offspring's net disposable income and that there is an accumulative education function. Becker and Tomes (1979) used this set of hypotheses to analyse the equilibrium distribution of income and intergenerational mobility. They labeled the approach based on the offspring's wealth by the term "quality of the children". They claimed that the implications in terms of income distribution of this

[3]Higher than an endogenous threshold.

[4]In a slightly different setting Schneider (2010) develops a model with diversity of occupations and parent-child dynamic transmission through a Markovian process: the *talent* of a child depends on the talent of her parents through a Markov process)

approach are similar to those of Barro's (1974) "dynastic altruism" approach, in which altruists care about their offsring's utility. Lambrecht et al (2005) and Lambrecht et al (2006) studied the properties of fiscal policies under this approach, which they labeled "family altruism". They find less clear cut conclusions: pay-as-you-go policies are neutral but public debt is not. The family altruism approach enables to study the transitional dynamics of physical and/or human capital[5]. As an altruistic bequest motive, it is also preferable to Andreoni's (1989) joy-of-giving or warm-glow motive because the latter is insensitive to the economic situation of the beneficiaries of transfers.

To summarize, this paper is based on the threefold assumption of (i) family altruism (ii) accumulative human capital and (iii) the existence of two distinct occupational choices (skilled and unskilled jobs). Among skilled agents, there is room for heterogeneity in human capital and ncome. In that sense it combines (i) the neoclassical approach which sees human capital as efficiency units of labor whose individual endowments vary across the skilled workers and (ii) the approach which emphasizes the role of indivisibilities in occupational choice[6].

We study the equilibrium paths along which human capital is accumulated differently across each generation's family. The main results are the following. First, we prove that there exists a unique intertemporal equilibrium path starting from any initial distribution of human capital. Secondly, we establish necessary and sufficient conditions for the existence of stationary equilibrium paths compatible with inequality in income among families and show that there exist a multiplicity of steady states. At these steady states, one share of the population permanently supply skilled labor while the remaining share permanently has zero human capital across generations and supplies unskilled labor. Then the analysis of convergence shows that any equilibrium path converges to a steady state equilibrium with persistent inequality between skilled and unskilled agents and equality among the skilled workers. We numerically illustrate that the long run relationship between inequality and output is ambiguous. Finally, we propose an endogenous version of the model by assuming that the productivity of unskilled labor benefits from the accumulation of human capital. In this version of the model, it is shown that the relationship between inequality and growth of output is also ambiguous.

The paper starts with the presentation of the model in Section 2. Then the competitive equilibrium is analyzed in Section 3. Section 4 studies steady state equilibria. Section 5 establishes the convergence of equilibrium paths to a steady state equilibrium and discusses the dynamics of inequality. In Section 6 we discuss the long run inequality-output relationship. Section 7 develops an endogenous-growth version of our model. Section 8 concludes.

[5]We confine our analysis to human capital only.

[6]Indivisibilities may also come from the educational system like in Chusseau and Hellier (2010).

2 The model

2.1 The firms

We consider a closed economy in which, at each time t, the output of the representative firm, Y_t, is determined by a Cobb-Douglas production function:

$$Y_t = (H_t)^{\alpha}(L_t)^{1-\alpha},$$

where H_t is the supply of skilled labor and L_t is the supply of unskilled labor. The wage rates of unskilled labor and human capital, w_t^L and w_t^H, are determined by their marginal products:

$$w_t^H = \alpha \left(\frac{L_t}{H_t}\right)^{1-\alpha}, \; w_t^L = (1-\alpha)\left(\frac{H_t}{L_t}\right)^{\alpha}.$$

Output is either consumed or spent on education.

2.2 The households

The economy is modelled as a sequence of successive generations. Each agent lives for one time period and has one offspring. The set of dynasties is the interval $[0,1]$. Each dynasty is denoted by the variable i. Each agent is endowed with one unit of unskilled labor force that requires no higher education and a subset of agents are also endowed with some amount of human capital. We call agents with positive endowment of human capital "educated" agents and those with zero endowment of human capital "uneducated" agents.

In dynasty i at time t, the agent's human capital is denoted by $h_t(i)$. This level of human capital depends on her parent's human capital, $h_{t-1}(i)$ and on the amount her parent spent for higher education, $e_{t-1}(i)$. We assume that this dependence is described as follows:

$$h_t(i) = e_{t-1}(i)^{\kappa}(h_{t-1}(i)+1)^{1-\kappa}, \quad 0<\kappa<1, \quad \forall t \geq 0. \tag{1}$$

If the parent of an individual spent nothing on her education, the human capital of this individual is nil.

Consider the individual belonging to dynasty i and living in period t. During this period, she supplies inelastically either unskilled labor or human capital. If $w_t^L > w_t^H h_t(i)$, she suppies one unit of unskilled labor. If $w_t^L < w_t^H h_t(i)$, she supplies $h_t(i)$ units of human capital. If $w_t^L = w_t^H h_t(i)$, she is indifferent in this respect. Thus, her total income is

$$\omega_t(i) = \max\{w_t^L, w_t^H h_t(i)\}, \quad \forall t \geq 0. \tag{2}$$

Total income is divided between consumption $c_t(i)$ and educational expenditure for her offspring $e_t(i)$. This educational expenditure is motivated by family altruism (see Lambrecht *et al.* 2005 and Lambrecht *et al.* 2006), i.e. by the concern for the offspring's total income $\omega_{t+1}(i)$ defined by (2) at time $t+1$. According to (1) at time $t+1$, spending $e_t(i)$ on the offspring's education determines the latter's human capital, and thus her total income.

The individual' preferences are defined over consumption $c_t(i)$ and her offspring's expected total income $\omega_{t+1}(i)$. They are represented by the following log-linear utility function: $\ln c_t(i) + \ln \omega_{t+1}(i)$. The individual maximizes her utility function under her budget constraints considering current wage and expectations on next period wages as given. We state this problem as follows

$$\max_{c_t(i) \geq 0, e_t(i) \geq 0} \ln c_t(i) + \ln \omega_{t+1}(i)$$

under the following constraints:

$$e_t(i) + c_t(i) = \omega_t(i), \tag{3}$$

$$\omega_{t+1}(i) = \max\{w_{t+1}^L, w_{t+1}^H h_{t+1}(i)\}, \tag{4}$$

$$h_{t+1}(i) = e_t(i)^\kappa (h_t(i) + 1)^{1-\kappa}. \tag{5}$$

The solution of this problem is characterized by the optimal educational expenditure $e_t(i)$ and can be solved in two steps. First we solve the two sub-problems defined by the two alternatives of the offspring's income. Second we select the solution leading to the highest utility. The optimal decision to invest in the offspring's education is described in the following proposition :

Proposition 1. *The optimal educational expenditure $e_t(i)$ and the resulting offspring's human capital $h_{t+1}(i)$ are determined as follows:*

- *they are respectively characterized by the following functions: $\hat{e}^H\big(\omega_t(i), h_t(i)\big) = \frac{\kappa}{1+\kappa}\omega_t(i)$ and $\hat{h}^H\big(\omega_t(i), h_t(i)\big) = \gamma\big(\omega_t(i)\big)^\kappa \big(1 + h_t(i)\big)^{1-\kappa}$, with $\gamma = (\frac{\kappa}{1+\kappa})^\kappa$ if*

$$\frac{\hat{h}^H\big(\omega_t(i), h_t(i)\big)}{1+\kappa} > \frac{w_{t+1}^L}{w_{t+1}^H}; \tag{6}$$

- *they are both equal to zero if*

$$\frac{\hat{h}^H\big(\omega_t(i), h_t(i)\big)}{1+\kappa} < \frac{w_{t+1}^L}{w_{t+1}^H} \tag{7}$$

- *there are two solutions, either $e_t(i) = 0$ and $h_{t+1} = 0$ or $e_t(i) = \hat{e}^H\big(\omega_t(i), h_t(i)\big)$ and $h_{t+1}(i) = \hat{h}^H\big(\omega_t(i), h_t(i)\big)$ if*

$$\frac{\hat{h}^H\big(\omega_t(i), h_t(i)\big)}{1+\kappa} = \frac{w_{t+1}^L}{w_{t+1}^H}; \tag{8}$$

Proof : see appendix A

This proposition reads that the parent's decision to invest in her offspring's education depends on the latter's occupational choice. If at time $t+1$ dynasty i individual supplies unskilled labor on the labor market, then at time t the parent's expenditure on education is nil and all her income $\omega_t(i)$ is spent on consumption. Hence the offspring's human capital, $h_{t+1}(i)$, is also nil. On the contrary, if at time $t+1$ dynasty i individual supplies human capital on the labor market, then the parent's expenditure on education is equal to the fraction $\frac{\kappa}{1+\kappa}$ of her income $\omega_t(i)$. The rest of the income, $\frac{1}{1+\kappa}\omega_t(i)$, is spent on consumption. Thus, the offspring's human capital is equal to $\hat{h}^H\big(\omega_t(i), h_t(i)\big)$.

At each time t it is convenient to order the set of dynasties in a way such that the function $h_t(\cdot)$ defined on the interval $[0,1]$ and describing the distribution of human capital across dynasties be non-decreasing. At the same time it follows from (40) that if for some i and j, $h_t(i) \geq h_t(j)$ and $\omega_t(i) \geq \omega_t(j)$, then $\hat{h}^H\big(\omega_t(i), h_t(i)\big) \geq \hat{h}^H\big(\omega_t(j), h_t(j)\big)$. This give us the opportunity to order the set of dynasties at time 0 and restrict our consideration to paths of the economy such that all functions $h_t(\cdot)$ are non-decreasing at the initial order.

3 Competitive equilibrium

To study the general equilibrium of this economy we proceed in two steps[7]. We first study the time t temporary equilibrium in subsection 3.1 given past variables and expectations of the future. In subsection 3.2, we then describe the intertemporal equilibrium with perfect foresight as a sequence of temporary equilibria with some adequate initial conditions and rule for expectations.

3.1 Time t temporary equilibrium

To define the temporary equilibrium at time t, we consider all past variables and expectations of the future as given. The latter are expectations of the next period wages w_{t+1}^L and w_{t+1}^H and the former are the time $t-1$ human capital levels, $h_{t-1}(i)$ $\forall i \in [0,1]$, total incomes $\omega_{t-1}(i) = \max\{w_{t-1}^L, w_{t-1}^H h_{t-1}(i)\}$ $\forall i \in [0,1]$, and educational spendings $e_{t-1}(i)$ $\forall i \in [0,1]$. Since these given past variables determine time t human capital levels $h_t(i)$ $\forall i \in [0,1]$, we can say that these levels are completely pre-determined by time $t-1$ decisions. To be more precise, all we need to know to construct the time t temporary equilibrium is the function $h_t(\cdot)$.

Let us assume that the function $h_t(\cdot)$ is non-decreasing and upper semi-continuous[8] and that $\int_0^1 h_t(i)di > 0$.

A time t *temporary equilibrium* is defined by a quadruple of functions $\{\omega_t(\cdot), c_t(\cdot), e_t(\cdot), h_{t+1}(\cdot)\}$, defined on $[0,1]$, a pair of prices $\{w_t^L, w_t^H\}$, a triplet of aggregate variables $\{L_t, H_t, Y_t\}$ and a *pivotal* dynasty i_t^H satisfying the following requirements:

- all agents, households and firms, are at their optima;
- the set of dynasties supplying unskilled labor at time t is the interval $[0, i_t^H)$ and the set of dynasties supplying human capital is the interval $[i_t^H, 1]$;

[7]See Hicks (1939) or, more recently Grandmont (1983) for the articulation of these two steps.

[8]The assumption that the function $h_t(\cdot)$ is upper semi-continuous is made for technical reasons, to specify its values at points of discontinuity.

- all markets clear.

It should be noticed that the pivotal dynasty i_t^H dtermines the share of unskilled labor suppliers in the population. The share of human capital suppliers is respectively $1 - i_t^H$.

To make our presentation simple we also impose the following requirement on temporary equilibrium, which will not lead to any loss of generality:

- $h_{t+1}(i)$ is a non-decreasing upper semi-continuous function defined on $[0, 1]$.

We determine this equilibrium by writing the variables of the above-mentioned tuples as functions of past variables and expectations. To find a temporary equilibrium at time t it is sufficient to determine the pivotal dynasty i_t^H. Knowing it, one can easily determine the equilibrium values of all other variables.

In equilibrium, the supply of unskilled labor is equal to

$$L_t = \int_0^{i_t^H} 1di = i_t^H, \tag{9}$$

the supply of human capital is equal to

$$H_t = \int_{i_t^H}^1 h_t(i)di \tag{10}$$

and human capital and unskilled labor are paid at their marginal products:

$$w_t^H = \alpha \left(\frac{i_t^H}{\int_{i_t^H}^1 h_t(i)di} \right)^{1-\alpha}, \; w_t^L = (1-\alpha) \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right)^{\alpha}. \tag{11}$$

Therefore,

$$\frac{w_t^L}{w_t^H} = \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right).$$

Also, in equilibrium, we must have

$$w_t^H h_t(i) \leq w_t^L, \; 0 \leq i < i_t^H,$$

and

$$w_t^H h_t(i) \geq w_t^L, \; i_t^H \leq i \leq 1,$$

or, equivalently,

$$h_t(i) \leq \frac{w_t^L}{w_t^H}, \; 0 \leq i < i_t^H,$$

and

$$h_t(i) \geq \frac{w_t^L}{w_t^H}, \; i_t^H \leq i \leq 1.$$

Therefore, given the time t human capital levels $h_t(i) \forall\ i \in [0,1]$, the time t equilibrium pivotal dynasty, i_t^H, is determined by the following conditions:

$$h_t(i) \leq \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right),\ 0 \leq i < i_t^H, \tag{12}$$

$$h_t(i) \geq \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right),\ i_t^H \leq i \leq 1. \tag{13}$$

The integral $\int_{i^H}^1 h_t(i)di$ is a continuous and non-increasing function of i^H because the function $h_t(\cdot)$ is bounded and non-negative. Therefore, $\frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i^H}^1 h_t(i)di}{i^H} \right)$ is a continuous decreasing function of i^H defined on the interval $(0,1]$. It tends to ∞ as i^H goes to 0, and its value at $i^H = 1$ is 0.

At the same time $h_t(i)$ is a non-decreasing function of i. Therefore, to find the pivotal dynasty it is sufficient to "solve" the following equation in i^H:

$$h_t(i^H) = \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i^H}^1 h_t(i)di}{i^H} \right). \tag{14}$$

If the solution to equation (14) exists, the time t equilibrium pivotal dynasty coincides with this solution. In this case this dynasty is the one with human capital just equal to the ratio between unskilled labor wage rate and human capital wage rate and hence for this dynasty supplying unskilled labor will result in the same income as supplying human capital:

$$w_t^L = w_t^H h_t(i_t^H).$$

A solution to (14) may not exist because $h_t(i)$ is not necessarily continuous. But even in the case of non-existence there is a unique i_t^H satisfying (12)-(13). It is given by

$$i_t^H = \min \left\{ i^H \in [0,1] \mid h_t(i^H) \geq \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i^H}^1 h_t(i)di}{i^H} \right) \right\}.$$

The above minimum exists because, as noted, $\frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i^H}^1 h_t(i)di}{i^H} \right)$ is a continuous function of i^H and the function $h_t(\cdot)$ is upper semi-continuous by assumption, and hence the set $\left\{ i^H \in [0,1] \mid h_t(i^H) \geq \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i^H}^1 h_t(i)di}{i^H} \right) \right\}$ is closed.

It should be noted that $0 < i_t^H < 1$ because $h_t(i) < \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right)$ for all sufficiently small $i > 0$ and $h_t(i) > \frac{(1-\alpha)}{\alpha} \left(\frac{\int_{i_t^H}^1 h_t(i)di}{i_t^H} \right)$ for i in the neighborhood of 1.

Knowing i_t^H, we get H_t, L_t, w_t^H and w_t^L from (9)–(11). Also we are able to determine the total income of all households:

$$\omega_t(i) = \begin{cases} w_t^L, & 0 \leq i < i_t^H, \\ w_t^H h_t(i), & i_t^H \leq i \leq 1. \end{cases}$$

Since $h_t(\cdot)$ is a non-decreasing upper semi-continuous function and $w_t^H h_t(i_t^H) \geq w_t^L$, the function $\omega_t(\cdot)$ is also non-decreasing and upper semi-continuous.

With the pairs $\{h_t(i), \omega_t(i)\}$ $\forall i$, we can now determine the time t equilibrium educational expenditures, $e_t(i)$, i.e. the optimal educational expenditures at equilibrium prices given expectations on next period wage rates w_{t+1}^L and w_{t+1}^H. Once this variable is determined, it will give us the next period distribution of human capital, the $h_{t+1}(i)$'s. Here we should notice that the functions $e_t(\cdot)$ and $h_{t+1}(\cdot)$ are not necessarily uniquely determined, because for i satisfying (8), $e_t(i)$ is equal to either 0 or $\hat{e}^H\big(\omega_t(i), h_t(i)\big)$. However, this non-uniqueness plays no role in our model, because, as will be shown in the next subsection, in intertemporal equilibrium non-uniqueness does not appear.

For short, in what follows we identify any temporary equilibrium at time t with the couple $\{i_t^H, h_{t+1}(\cdot)\}$.

3.2 The intertemporal equilibrium with perfect foresight

Suppose we are given an initial state of the economy represented by a non-decreasing upper semi-continuous function $h_0(\cdot)$ showing the distribution of human capital across dynasties at the initial time. We assume that $\int_0^1 h_0(i)di > 0$ and define an *intertemporal equilibrium* path $\{i_t^H, h_{t+1}(\cdot)\}_{t=0}^{\infty}$ starting from $h_0(\cdot)$ as a sequence of temporary equilibria, such that at each time t each dynasty has perfect foresight, that is, correctly anticipate time $t+1$ wage rates.

We proceed recursively as follows. Given $h_0(\cdot)$, we construct i_0^H as described in the previous section. After that, for each $t = 0, 1, ...$, given $h_t(\cdot)$ and i_t^H, we construct $h_{t+1}(\cdot)$ and i_{t+1}^H simultaneously in a way consistent with the procedure described in the previous section.

Suppose that the function $h_t(\cdot)$ is given, that it is non-decreasing and upper semi-continuous and that $\int_0^1 h_t(i)di > 0$. Suppose further that i_t^H is found as described in the previous section (if $t = 0$), or at step $t-1$ (if $t > 0$) and that w_t^L, w_t^H and $\omega_t(\cdot)$ are found as described in the previous section. To find the function $h_{t+1}(\cdot)$, we start with using the time $t+1$ human capital function associated with positive investment in education given by equation (40) in section 2.2, namely $\hat{h}^H\big(\omega_t(i), h_t(i)\big)$. Since the analysis will focus on the pivotal dynasty and all the arguments of this function depend on i, we re-write it as a function $\tilde{h}_{t+1}$ of i.

Namely, let the function $\tilde{h}_{t+1} : [0,1] \to \mathbb{R}_+$ be defined by

$$\tilde{h}_{t+1}(i) = \hat{h}^H\big(\omega_t(i), h_t(i)\big) \; \left(= \gamma \left[\omega_t(i)\right]^{\kappa} \big(1 + h_t(i)\big)^{1-\kappa}\right).$$

Since $\omega_t(\cdot)$ and $h_t(\cdot)$ are non-decreasing upper semi-continuous functions, $\tilde{h}_{t+1}(\cdot)$ is also a non-decreasing upper semi-continuous function. Moreover,

$$\tilde{h}_{t+1}(i) = \gamma \left[w_t^L\right]^{\kappa}, \; 0 \leq i < i_t^H, \tag{15}$$

$$\tilde{h}_{t+1}(i) = \gamma \left[w_t^H\right]^{\kappa} \psi(h_t(i)), \; i_t^H \leq i \leq 1, \tag{16}$$

where the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\psi(h) = h^{\kappa}(h+1)^{1-\kappa}. \tag{17}$$

Let further the function $\mathcal{H}_{t+1} : [0,1] \to \mathbb{R}_+$ be defined by

$$\mathcal{H}_{t+1}(j) = \int_j^1 \tilde{h}_{t+1}(i)di. \tag{18}$$

This function is continuous and decreasing because the function $\tilde{h}_{t+1}(\cdot)$ is bounded and non-negative. It shows the dependence of the aggregate supply of human capital on the pivotal dynasty. In equilibrium at time $t+1$ the ratio of the wage rates of unskilled labor and human capital is endogenously determined by the marginal productivities of these inputs, which in turn is determined by the relative masses of these inputs. So we have:

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(i^H_{t+1})}{i^H_{t+1}} = \frac{w^L_{t+1}}{w^H_{t+1}}$$

and, at the same time, from Proposition 1 and the definition of the function $\tilde{h}_{t+1}(\cdot)$

$$\frac{\tilde{h}_{t+1}(i)}{1+\kappa} \le \frac{w^L_{t+1}}{w^H_{t+1}},\ 0 \le i < i^H_{t+1}, \tag{19}$$

$$\frac{\tilde{h}_{t+1}(i)}{1+\kappa} \ge \frac{w^L_{t+1}}{w^H_{t+1}},\ i^H_{t+1} \le i \le 1. \tag{20}$$

Whereas $\frac{\mathcal{H}_{t+1}(j)}{j}$ is a decreasing function of j, $\tilde{h}_{t+1}(i)$ is a non-decreasing function. Moreover,

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} > \frac{\tilde{h}_{t+1}(j)}{1+\kappa}$$

for all sufficiently small $j > 0$ and

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} < \frac{\tilde{h}_{t+1}(j)}{1+\kappa}$$

for all j sufficiently close to 1.

To find the time $t+1$ equilibrium pivotal dynasty i^H_{t+1}, it is sufficient to "solve" the following equation in j:

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} = \frac{\tilde{h}_{t+1}(j)}{1+\kappa}. \tag{21}$$

If this equation has a solution, it is unique. Since $\tilde{h}_{t+1}(\cdot)$ may be discontinuous, the non-existence of a solution to equation (21) is possible. But even if (21) has no solution, the time $t+1$ equilibrium pivotal dynasty i^H_{t+1} is uniquely determined by the following conditions:

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} > \frac{\tilde{h}_{t+1}(j)}{1+\kappa},\ 0 \le j < i^H_{t+1}, \tag{22}$$

$$\frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} \le \frac{\tilde{h}_{t+1}(j)}{1+\kappa},\ i^H_{t+1} \le j \le 1. \tag{23}$$

It is given by

$$i^H_{t+1} = \min\left\{ j \in [0,1] \mid \frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} \leq \frac{\tilde{h}_{t+1}(j)}{1+\kappa} \right\}.$$

The above minimum exists because the function $\mathcal{H}_{t+1}(\cdot)$ is continuous and the function $\tilde{h}_{t+1}(\cdot)$ is upper semi-continuous, and hence the set $\left\{ j \in [0,1] \mid \frac{1-\alpha}{\alpha}\frac{\mathcal{H}_{t+1}(j)}{j} \leq \frac{\tilde{h}_{t+1}(j)}{1+\kappa} \right\}$ is closed.

Now we determine $h_{t+1}(\cdot)$ as follows:

$$h_{t+1}(i) = 0,\ 0 \leq i < i^H_{t+1},$$

$$h_{t+1}(i) = \tilde{h}_{t+1}(i),\ i^H_{t+1} \leq i \leq 1. \tag{24}$$

Thus, we have proved

Theorem 1. *For any initial state $h_0(\cdot)$ there is a unique intertemporal equilibrium path starting from this initial state.*

It should be noticed that, unlike temporary equilibrium, non-uniqueness of equilibria does not appear in intertemporal equilibrium. This is because in the definition of temporary equilibrium at time t agents take the wage rates at time $t+1$ as given, whereas in intertemporal equilibrium they are determined endogenously. Also it is noteworthy that in intertemporal equilibrium an agent spend a strictly positive fraction of her income on education if and only if her offspring will be human capital supplier on the labor market.

4 Steady-state equilibria

Steady-state equilibria are characterized by the feature that the wage rates and the shares of educated agents supplying human capital on the labor market and uneducated agents supplying unskilled labor are constant over time and that inside each dynasty children find themselves in the same position as their parents.

At a steady-state equilibrium, the amount of human capital, h^*, supplied by an agent from an educated dynasty only depends on the wage paid to one unit of human capital, w^{H*}, because h^* is the solution to the following equation: $h = \gamma(w^{H*})^\kappa\psi(h)$. Hence, at a steady-state equilibrium all educated dynasties supply the same amount of it. Therefore we can define steady-state equilibria as follows.

A couple (i^{H*}, h^*), $i^{H*} \in (0,1)$, $h^* > 0$, is called *a steady-state equilibrium* if the sequence $\{i_t, h_{t+1}(\cdot)\}_{t=0}^\infty$ given by

$$i_t = i^{H*},\ t = 0,1,2,\ldots,$$

$$h_{t+1}(i) = 0,\ 0 \leq i < i^{H*},\ t = 0,1,2,\ldots,$$

$$h_{t+1}(i) = h^*,\ i^{H*} \leq i \leq 1,\ t = 0,1,2,\ldots,$$

is an equilibrium path starting from $h_0(\cdot)$ defined as follows:

$$h_0(i) = 0,\ 0 \leq i < i^{H*},$$

$$h_0(i) = h^*, \; i^{H*} \leq i \leq 1.$$

It follows from (10) and (11) that at any steady state equilibrium (i^{H*}, h^*) the total supply of human capital, H^* and the wage rates of unskilled labor and human capital, w^{L*} and w^{H*}, are given as follows:

$$H^* = (1 - i^{H*})h^*,$$

$$w^{H*} = \alpha \left(\frac{i^{H*}}{H^*} \right)^{1-\alpha} = \alpha \left(\frac{i^{H*}}{(1 - i^{H*})h^*} \right)^{1-\alpha}, \tag{25}$$

$$w^{L*} = (1 - \alpha) \left(\frac{H^*}{i^{H*}} \right)^{\alpha} = (1 - \alpha) \left(\frac{(1 - i^{H*})h^*}{i^{H*}} \right)^{\alpha}. \tag{26}$$

In a steady state equilibrium the ratio of unskilled labor wage to human capital wage is given by :

$$\frac{w^{L*}}{w^{H*}} = \frac{1 - \alpha}{\alpha} \frac{(1 - i^{H*})h^*}{i^{H*}}, \tag{27}$$

which is a decreasing function of i^{H*} and an increasing function of h^*. As a measure of income inequality we use the *skill premium* P^*. It is defined as the proportion of the wage earned by an educated individual to the wage of an unskilled individual:

$$P^* = \frac{w^{H*}h^*}{w^{L*}}.$$

It follows from (27) that on the set of steady-state equilibria the skill premium is an increasing function of i^{H*}:

$$P^* = \frac{\alpha}{1 - \alpha} \frac{i^{H*}}{1 - i^{H*}}.$$

Let (i^{H*}, h^*) be a steady-state equilibrium and $\{i_t, h_{t+1}(\cdot)\}_{t=0}^{\infty}$ be the corresponding equilibrium path. It follows from (15)-(16) that for this path,

$$\tilde{h}_{t+1}(i) = \begin{cases} \gamma(w^{L*})^{\kappa}, & 0 \leq i < i^{H*}, \\ \gamma(w^{H*})^{\kappa}\psi(h^*), & i^{H*} \leq i \leq 1. \end{cases}$$

Therefore at a steady state conditions (19) and (20) respectively become

$$\frac{\gamma(w^{L*})^{\kappa}}{1 + \kappa} \leq \frac{w^{L*}}{w^{H*}} \tag{28}$$

and

$$\frac{\gamma(w^{H*})^{\kappa}\psi(h^*)}{1 + \kappa} \geq \frac{w^{L*}}{w^{H*}}. \tag{29}$$

The first of these inequalities means that the uneducated agents have no incentives to spend on the education of their offsprings and the second that the educated agents have such incentives. It is also clear that

$$h^* = \gamma(w^{H*})^{\kappa}\psi(h^*). \tag{30}$$

One can easily prove the following proposition.

Proposition 2. *A couple (i^{H*}, h^*), $0 < i^{H*} < 1$, $h^* > 0$, is a steady-state equilibrium if and only if for w^{H*} and w^{L*} given by (25) and (26) respectively, (28)-(30) hold true.*

Let us now describe the relationship between the share of uneducated agents in the population, i^{H*}, and the human capital accumulated by an educated agent in a steady-state equilibrium, h^*. It is reasonable to conjecture that this relationship is increasing because a higher share of uneducated agents can lead to a larger skill premium and wages of educated individuals and hence to higher individual educational expenditures. The following lemma says that this conjecture is true.

Lemma 2. *There is a smooth increasing function $\chi : (0,1) \to \mathbb{R}_+$ and numbers L_1 and L_2, $0 < L_2 < L_1 \leq 1$, such that for any $i^* \in (0,1)$ and for w^{H*} and w^{L*} given by (25) and (26) respectively,*

(30) is equivalent to

$$h^* = \chi(i^{H*}), \tag{31}$$

(29) is equivalent to $i^{H*} \geq L_2$,

(28) is equivalent to $i^{H*} \leq L_1$.

Proof: see appendix B

The following theorem describes the structure of steady-state equilibria. It directly follows from Proposition 2 and Lemma 2.

Theorem 3. *There exists a smooth increasing function $\chi : (0,1) \to \mathbb{R}_+$ and numbers L_1 and L_2, $0 < L_2 < L_1 \leq 1$, such that a couple (i^{H*}, h^*) is a steady-state equilibrium if and only if either*

$$L_2 \leq i^{H*} \leq L_1 \ (\textit{if } L_1 < 1)$$

or

$$L_2 \leq i^{H*} < L_1 \ (\textit{if } L_1 = 1)$$

and

$$h^* = \chi(i^{H*}).$$

According to Theorem 3, the set of steady-state equilibria is a continuum which can be parameterized by the share of uneducated agents in the population[9].

Another interesting parametrization of the set of steady-state equilibria is that by the skill premium. This parametrization can help us to explain why all equilibrium values of the share of uneducated agents lies in the interval $[L_2, L_1]$. If $i^{H*} < L_2$, then the couple $(i^{H*}, \chi(i^{H*}))$ is not a steady-state equilibrium because the wage rate of unskilled labor is so high and the skill premium is so small that even the educated parents have no incentives to spend a positive fraction of their incomes on the education of their children. If $L_2 \leq i^{H*} \leq L_1$, then the skill premium is such that the educated individuals prefer to see their children educated while the uneducated agents find it too expensive to spend money on the education of their children. Finally, if $i^{H*} > L_1$, the couple $(i^{H*}, \chi(i^{H*}))$ is not a steady-state equilibrium because the wage rate of unskilled labor is so small and the skill premium is so high that even the uneducated individuals are ready to spend money on the education of their children.

[9]In Ray (2006), the set of steady state equilibria also is a continuum.

5 Convergence of equilibrium paths

In this section, we first shall prove that starting from any initial state, characterized by an arbitrary distribution of human capital, the economy converges to a steady state and we shall provide a graphical illustration of the convergence process.

The following theorem reads that any equilibrium path converge to a steady-state equilibrium and that the number of uneducated agents does not increase over time (except, perhaps, at time $t = 1$).

Theorem 4. *For any equilibrium path $\{i_t^H, h_{t+1}(\cdot)\}_{t=0}^{\infty}$ the sequence $\{i_t^H\}_{t=1}^{\infty}$ is non-increasing (it may be that $i_1^H > i_0^H$) and there is a steady-state equilibrium (i^{H*}, h^*) such that*

$$i_t^H \longrightarrow_{t\to\infty} i^{H*},$$

$$h_t(i) \longrightarrow_{t\to\infty} h^*,\ \ i^{H*} \leq i \leq 1.$$

Proof: see appendix C

Traditionnally in models of economic growth with a representative agent or with homegeneous generations, there is a unique steady state equilibrium (see for example the Ramsey model) or a finite number of steady states (for example the overlapping-generation model with production). In these models the question of convergence boils down to connecting initial states with the unique or the finite number of steady state. In growth models with heterogeneous agents (Banerjee and Newman (1993), Galor and Zeira (1993), Freeman (1996), Ray (2006) and Borissov and Lambrecht (2009)), there is a continuum of steady states and the question of convergence is more complex. On the one hand we cannot determine to which steady state the economy will converge if it starts from a given initial state. On the other hand for any steady state there is a continuum of initial distributions of human capital that would lead to the convergence to this steady state.

We develop an example of transition to shed light on the process of human capital accumulation in the presence of indivisibilities in occupations. Figures 1-5 show the economy converges to a steady state starting from an arbitrary distribution of human capital. We shall discuss how the shares of skilled and unskilled workers evolve across time, how dynasties respond to incentives given by the evolving skill premium and shed light on the long run division of society into these two categories.

Figure 1 shows the distribution of generation 0 human capital levels. Members of generation 0 are ordered according to their human capital. We give ourselves an initial distribution with positive human capital for all individuals with the exception of the origin.

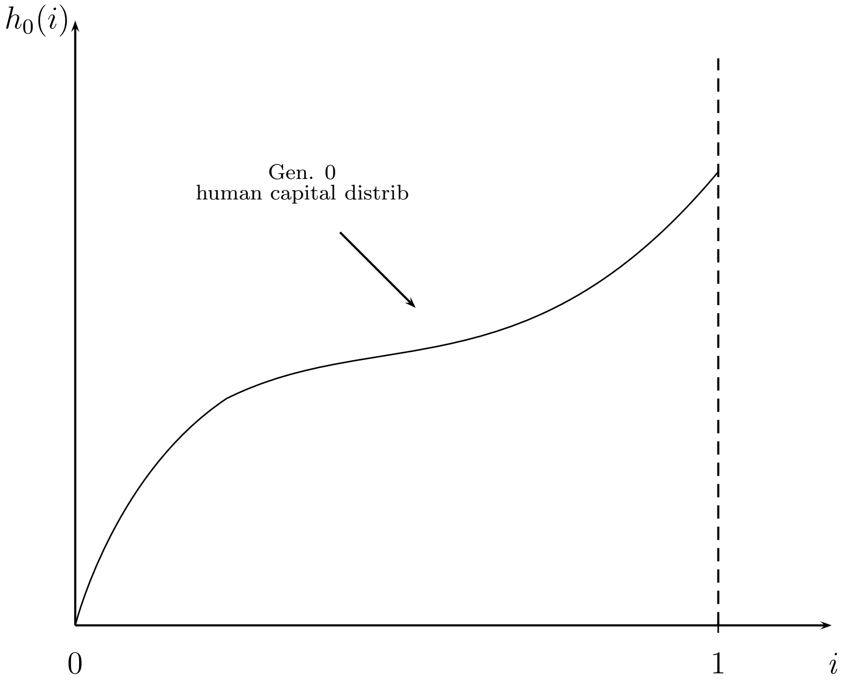

Figure 1: Generation 0 human capital distribution

Figure 2 shows how the initial generation, while having positive human capital, splits into two groups as far as their occupation is concerned. The production process implies both skilled and unskilled labor. The unskilled occupations can be taken by individuals with positive human capital but the latter are not more productive in these occupation than individuals deprived of human capital. Generation 0 individuals with low human capital (i.e. those below i_0^H) choose to work as unskilled after comparing the respective total wages of skilled and unskilled labor. Indeed, given the market wage ratio, it is necessary to have a sufficiently high level of human capital to receive a higher total wage as a skilled than as an unskilled.

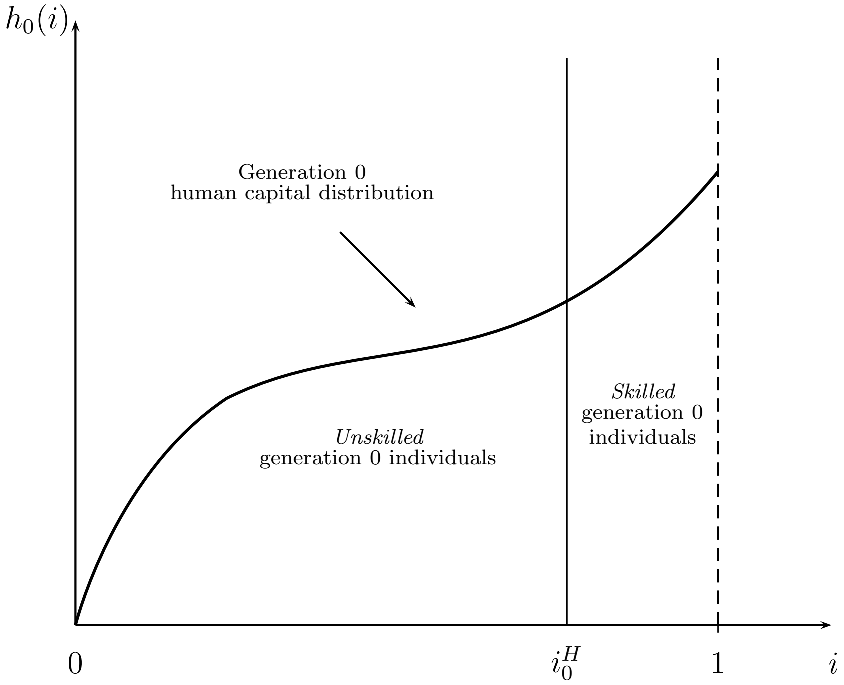

Figure 2: Division of generation 0 individuals between skilled and unskilled occupations

Generation 0 individuals decide whether to spend or not on the education of their offspring by comparing the respective expected total wages they would receive in the two types of occupation[10]. This is depicted in figure 3. In their reasoning, parents follow their altrusitic bequest motive and use the human capital production function. This function brings the influence of parents' human capital (the dotted line) into the picture.

[10]The difference between generation 0 choice of their own occupation and the choice of their offspring's occupation is that in the former case human capital is given, while in the latter it is a decision variable.

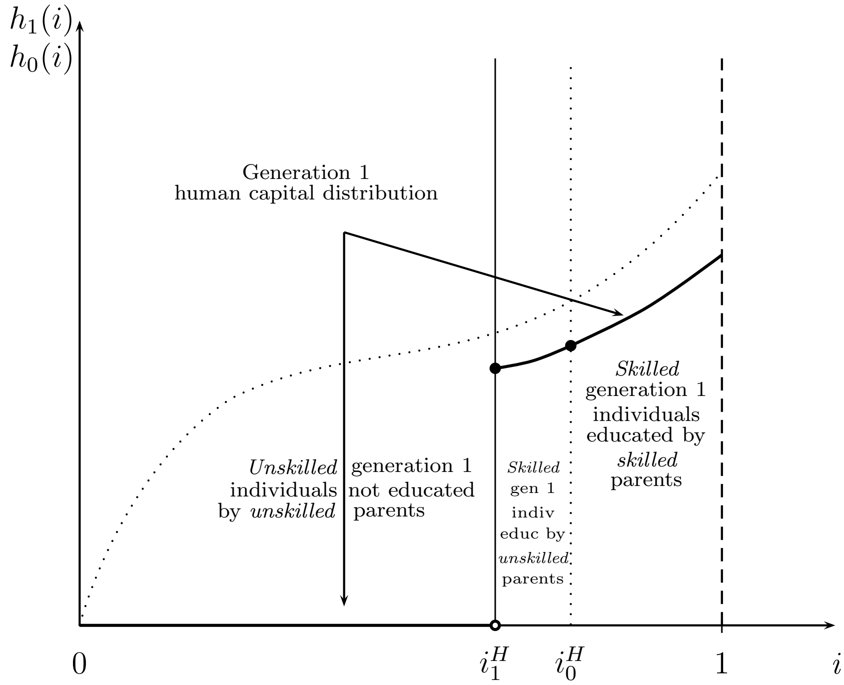

Figure 3: Generation 1 human capital distribution and switching dynasties

Two categories of families invest in human capital. First, generation 0 parents with positive human capital and occupying skilled jobs all spend positive amounts on their offspring's education. Second, a share of unskilled parents decides to invest in their children's education. In figure 3, these families lie in between the time $t = 0$ pivotal dynasty, i_0^H, and the time $t = 1$ one. As a result, time $t = 1$ individuals with positive human capital and occupying skilled jobs are represented by the two-piece curve on the right side of the graph. All other families are characterized as follows: parents are unskilled and children are not educated are occupy unskilled jobs.

The graph in Figure 4 shows the distribution of human capital at time $t = 2$. The period before, all parents have made their education investment decision by looking at the expected wage ratio and their own level of human capital. With respect to time $t = 1$ more unskilled families start to invest in their offspring's human capital.

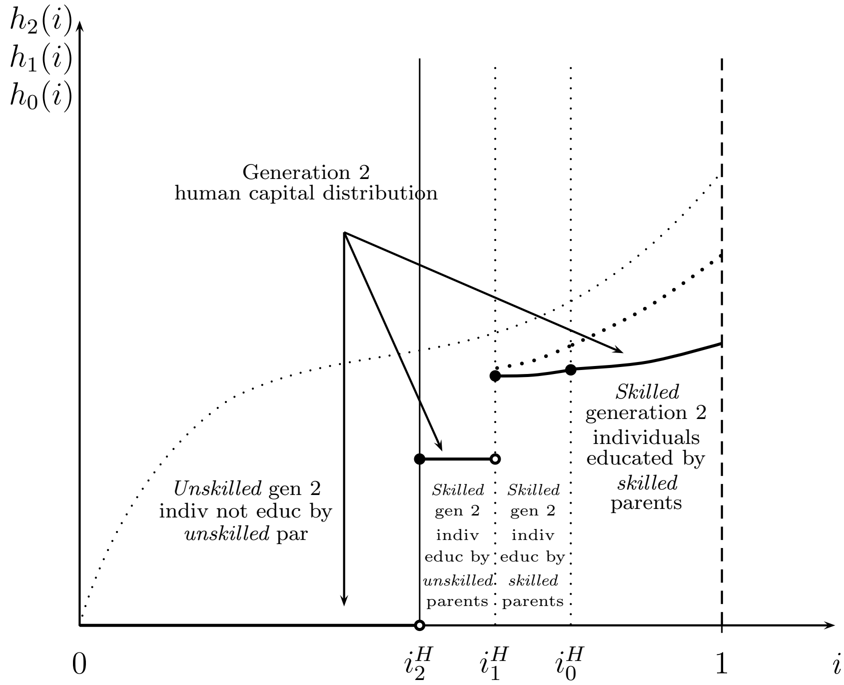

Figure 4: Generation 2 human capital distribution and switching dynasties at time $t = 2$

Graphs 3 and 4 show that there can be jumps in children's human capital for some unskilled dynasties. Hence these jumps lead to discontinuities in the distribution of human capital across dynasties at a given period. These jumps raise the following question: do children's wages and skill premia also jump for these switching dynasties? The model answers that there is a jump in children's total wages for switching dynasties: the total wages, $w_t^H h_t(i)$ of children in switching dynasties is larger than the total wage w_t^L of dynasties remaining unskilled. In graph 4 these dynasties are respectively located between i_2^H and i_1^H and between 0 and i_2^H. The mechanism behind this result is the following: both groups of unskilled dynasties (switching and not switching) have the same utility but parents in switching dynasties sacrifice some consumption ($\ln w_t^L > \ln[1/(1+\kappa)]w_t^L$) and compensate with higher utility from their children's income ($\ln w_{t+1}^L < \ln w_{t+1}^H h_{t+1}(i)$). In equilibrium if a smaller share of dynasties switched, the skilled labor supply would be marginally lower, the wage of skilled workers would be marginally higher and the utility under positive educational investment would be higher than utility with zero investment; as a consequence skilled labor supply would increase and the share of skilled workers in the population increase. Only in subsequent periods, will the utility of these switching dynasties be higher than those remaining unskilled.

The last figure (Figure 5) represents a possible steady state of the economy. As proposition 3 establishes, there is a continuum of steady state equilibria inside the interval $[L_2, L_1]$. In the graph, we choose one of these possible steady states and characterize it by the i^{H*} pivotal dynasty. At the chosen steady state equilibrium, skilled dynasties share the same steady level of human capital. This level is reproduced at every period. Unskilled dynasties steadily reproduce a zero level of human capital. Hence there is persistent long run inequality at the competititve steady state equilibrium.

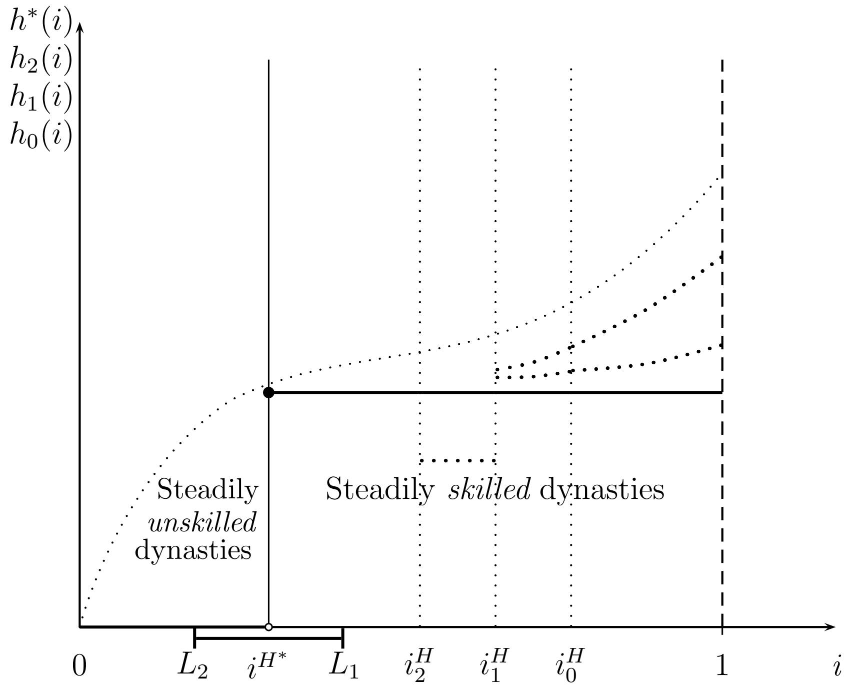

Figure 5: Steady state equilibrium human capital distribution

This graphical example sheds light on the dynamics of income inequality on the transition to the steady state. Two dynamic processes take place. They regard income inequality *between* skilled workers and unskilled workers (hereafter *between-inequality*) and income inequality *among* skilled workers (hereafter *within-inequality*).

Even if the economy starts with an equal distribution of human capital, income inequality between skilled and unskilled workers appears from the beginning. In subsequent period the share of skilled workers increases but this does not necessarily imply that between-inequality increases or decreases. Indeed this depends on the initial distribution of human capital.

As far as skilled workers are concerned, once a dynasty has joined the share of skilled workers, its human capital and wage income tend to converge towards a unique level because of diminishing returns in the education function. This process of equalization is perturbed by the newcomers dynasties. The latter enter the group of skilled dynasties at the bottom of the distribution of human capital. However when the process of switching from unskilled to skilled occupations stops, within-inequality monotonically decreases and tends to zero.

The key features of our model[11] make it possible to describe such a combination of the within- and between-inequality dynamics.

6 On the long run output-inequality nexus

Let us now consider the question of what is the relationship between inequality, measured by the skill premium, and output on the set of steady-state equilibria. The debate on this issue is not settled. The classical approach suggests that inequality stimulates capital accumulation and thus promotes economic growth, whereas the modern approach argues in contrast that for sufficiently wealthy economies equality stimulates investment in human capital and hence may enhance economic growth. To sketch the broad outlines of this relationship, it is sufficient to look at the dependence of output on i^{H*}, because, as noted above, the skill premium P^* is an increasing function of i^{H*}. The level of output, Y^*, corresponding to a steady-state equilibrium $(i^{H*}, \chi(i^{H*}))$ is

$$Y^* = [\underbrace{(1 - i^{H*})}_{-}\underbrace{\chi(i^{H*})}_{+}]^{\alpha}[\underbrace{i^{H*}}_{+}]^{1-\alpha}.$$

The signs under the components of Y^* indicate the positive or negative dependence of each component with respect to the steady pivotal dynasty i^{H*}. Remember that the lower the steady state pivotal dynasty, the higher the share of skilled workers $1 - i^{H*}$ and the lower the individual steady human capital $\chi(i^{H*})$. We can distinguish two features. First the dependence of unskilled labor with respect to the steady pivotal dynasty is monotonic and positive: as i^{H*} decreases, the contribution of of unskilled labor, $[i^{H*}]^{1-\alpha}$, decreases because there are less and less unskille workers. Second the dependence of total human capital, $(1-i^{H*})\chi(i^{H*})$, with respect to the steady pivotal dynasty is non monotonic. Let us contrast two polar situation. First consider a steady state in which the steady pivotal dynasty is close to the L_1 upper bound[12]. As a consequence the share of skilled workers $1 - i^{H*}$ is low but the individual steady human capital $\chi(i^{H*})$ is high. This case is one with high inequality between skilled and unskilled but the net effect on output depends on the dominating component, share of skilled or individual steady human capital. To the opposite, consider a steady state with low i^{H*} (close to the L_2 lower bound), inequality is lower but again the net effect on output is ambiguous.

It would be difficult to derive an analytical form of the function $[(1 - i)\chi(i)]^{\alpha}[i]^{1-\alpha}$. It is clear that the dependence of Y^* on i^{H*} is quite ambiguous. The shape of the graph of this function depends on the relative weight of skilled and unskilled labor and on the parameters of the model, α and κ. The higher α, the higher the elasticity of output with respect to human capital and therefore the higher the share of human capital income in

[11] These key features are : (i) human capital is a stock, (ii) there are indivisibilities in occupation. Other models either have one occupation and different human capital levels or have a fixed education costs and identical human capital of skilled workers.

[12] Remember that any steady state must lie in the (L_2, L_1) interval

output. The higher κ, the more efficient is parents' educational expenditure to produce the offspring's human capital. These parameters are key in the relationship between inequality and growth in the long run. Indeed, when κ is high, educational expenditure is so efficient to produce human capital that a lower steady level of spending is needed to sustain long run growth. Similarly, when α is high, the efficiency of human capital to produce output is so high that a lower level if it is needed to have high output

Our computational experiments[13] show that this dependence on the interval $[L_2, L_1]$ can be of an inverted U-shaped form or increasing.

On Fig. 1 we present L_1, L_2 and the graph of $[(1-i)\chi(i)]^{\alpha}[i]^{1-\alpha}$ on the segment $[0,1]$ at $\alpha = 0.4$, $\kappa = 0.5$. On $[L_2, L_1]$ the function $[(1-i)\chi(i)]^{\alpha}[i]^{1-\alpha}$ has an inverted U-shaped form.

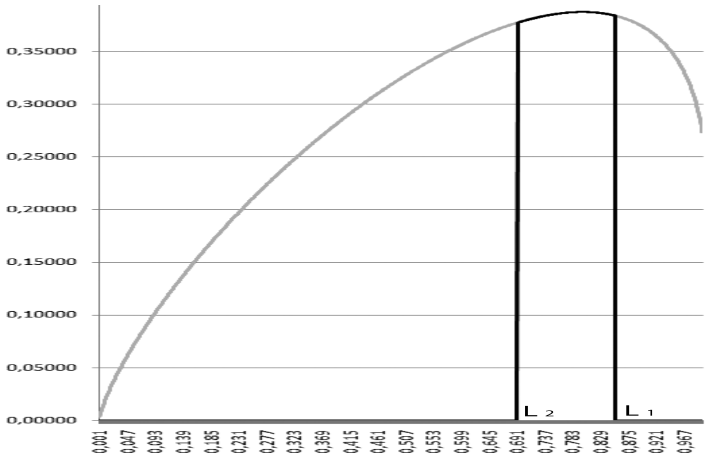

Figure 6

The interpretation of this pattern of dependence runs as follows. If we start from a very unequal steady state, i.e. close to L_1, and move towards a more equal one in which the share of skilled workers is higher, aggregate output increases because the positive effect playing through the increase of the share of skilled workers $1 - i^{H*}$ dominates the other two negative effects, through the share of unskilled i^{H*} and through the individual steady human capital $\chi(i^{H*})$. If we continue the move towards a more equal steady state, the two negative effects end up dominating the positive effect on Y^*

A different pattern of dependence appears if the elasticity of output w.r.t. human capital α is slightly lower and the efficiency of educational expenditure κ higher. On Fig. 2 we present L_1, L_2 and the graph of $\Gamma(i)$ on the segment $[0,1]$ at $\alpha = 0.3$, $\kappa = 0.96$. On $[L_2, L_1]$ the function $[(1-i)\chi(i)]^{\alpha}[i]^{1-\alpha}$ increasing.

[13]The computations have been performed with Matlab.

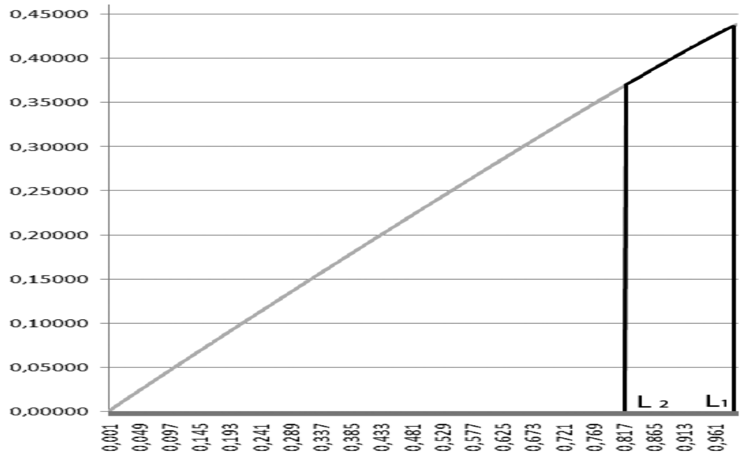

Figure 7

Again let us examine the effects of a move towards a more equal distribution of human capital on the level of output. As i^{H*} decreases the decrease in output induced by $\chi(i^{H*})^{\alpha}$ and $[i^{H*}]^{1-\alpha}$ always dominates the increase induced by $(1 - i^{H*})^{\alpha}$

7 Endogenous growth

In this section we propose an endogenous growth version of our model. To do this, we introduce an endogenously formed variable reflecting the state of technology at time t, A_t. An increase in its value leads (i) to a higher effectiveness of unskilled labor and (ii) promote accumulation of human capital. In its turn, the accumulation of human capital contribute to the growth of the value of this variable through a macroeconomic externality.

More precisely, our assumptions are as follows. The output at time t, Y_t, is given by

$$Y_t = H_t^{\alpha}(A_t L_t)^{1-\alpha}.$$

Therefore the wage earned by each agent supplying unskilled labor on the labor market is $w_t^L = (1-\alpha)H_t^{\alpha}(A_t L_t)^{1-\alpha}$.

The human capital of an agent of dynasty i at time t, $h_t(i)$, depends not only on the human capital of her parent, $h_{t-1}(i)$ and the amount of money the parent spent for her higher education, $e_{t-1}(i)$, but also on the state of technology at time $t-1$, A_{t-1}:

$$h_t(i) = e_{t-1}(i)^{\kappa}(h_{t-1}(i) + A_{t-1})^{1-\kappa}, \quad 0 < \kappa < 1. \tag{32}$$

As for the formation of A_t, we assume that

$$A_t = \Phi(H_{t-1}, A_{t-1}),$$

where $\Phi : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous homogeneous of degree one concave function.

Thus, the variable A_t *i*) shows the efficiency of unskilled labor and *ii*) plays the role of an input in the educational production function. Its value can grow over time through the accumulation of human capital.

The behavior of individuals is the same as in the case of the exogenous growth model with the only difference that (1) is replaced by (32). The temporal and intertemporal equilibria are also defined in practically the same way as above. The difference is that a temporal equilibrium at each time t is described not by a couple $\{i_t^H, h_{t+1}(\cdot)\}$, but by a triple $\{i_t^H, A_{t+1}, h_{t+1}(\cdot)\}$ satisfying the requirements formulated in Subsection 3.1 and the equation $A_{t+1} = \Phi(\int_0^1 h_t(i)di, A_t)$. Clearly, an initial state of an intertemporal equilibrium path is determined by a couple $\{A_0, h_0(\cdot)\}$. The existence of an intertemporal equilibrium paths is also proved in the same way.

As for *a steady-state equilibrium*, it is defined as a triple $\{i^{H*}, g^*, h^*\}$, $i^{H*} \in (0,1)$, $h^* > 0$, such that the sequence $\{i_t^H, A_{t+1}, h_{t+1}(\cdot)\}_{t=0}^{\infty}$ given by

$$i_t^H = i^{H*},\ t = 0, 1, 2, \ldots,$$

$$A_t = (1+g^*)^t,\ t = 0, 1, 2, \ldots,$$
$$h_{t+1}(i) = 0,\ 0 \le i < i^{H*},\ t = 0, 1, 2, \ldots,$$
$$h_{t+1}(i) = A_{t+1}h^*,\ i^{H*} \le i \le 1,\ t = 0, 1, 2, \ldots,$$

is an equilibrium path starting from the initial state $\{A_0, h_0(\cdot)\}$ given by $A_0 = 1$,

$$h_0(i) = 0,\ 0 \le i < i^{H*},$$

$$h_0(i) = h^*,\ i^{H*} \le i \le 1.$$

In this definition, g^* is an equilibrium balanced growth rate.

As in the case of exogenous growth, it is not difficult to show that *any equilibrium path* $\{i_t^H, A_{t+1}, h_{t+1}(\cdot)\}_{t=0}^{\infty}$ *converges to a steady-state equilibrium* in the following sense:

$$i_t^H \longrightarrow_{t\to\infty} i^{H*},$$

$$\frac{h_t(i)}{A_t} \longrightarrow_{t\to\infty} h^*,\ i^{H*} \le i \le 1,$$
$$\frac{A_{t+1}}{A_t} \longrightarrow_{t\to\infty} 1 + g^*.$$

The structure of the set of steady-state equilibria is described in the following theorem.

Theorem 5. *There is a smooth increasing function* $\chi : (0,1) \to \mathbb{R}_+$ *and numbers* L_1 *and* L_2*,* $0 < L_2 < L_1 \le 1$*, such that a triple* $(i^{H*}, 1+g^*, h^*)$ *is a steady-state equilibrium if and only if either*

$$L_2 \le i^{H*} \le L_1\ (\textit{if } L_1 < 1)$$

or

$$L_2 \le i^{H*} < L_1\ (\textit{if } L_1 = 1)$$

and

$$h^* = \chi(i^{H*}),\ 1 + g^* = \Phi((1-i^{H*})h^*, 1) \tag{33}$$

What does the endogenous growth version of our model tell about the nexus of income inequality and the rate of economic growth? In this version of the model, the nature of the relationship between income inequality and the rate of growth on the set of steady-state equilibria can be illustrated by the graph of the dependence of $(1 - i^{H*})\chi(i^{H*})$ on i^{H*} implied by (33). We already know that these two components of the rate of growth have opposite dependence on i^{H*}.

Similar results can be derived from a computational exercise as in the exogenous version.The parameter κ plays a same role. Our simulations show that on the interval $[L_2, L_1]$ the rate of growth of output can be a decreasing function of i^{H*} if κ is low (see Figure 8 with $\alpha = 0.4$, $\kappa = 0.5$); for intermediate values of κ, the pattern of relationship has an inverted U-shape form (see Figure 9 with $\alpha = 0.7$, $\kappa = 0.8$) and for high values of κ the rate of growth of output is an increasing function of i^{H*} (see Figure 10 with $\alpha = 0.3$, $\kappa = 0.96$). Figs. 8-10 present graphs of $(1 - i)\chi(i)$ on the segment $[0, 1]$ and indicate L_2 and L_1 for different values of parameters α and κ.

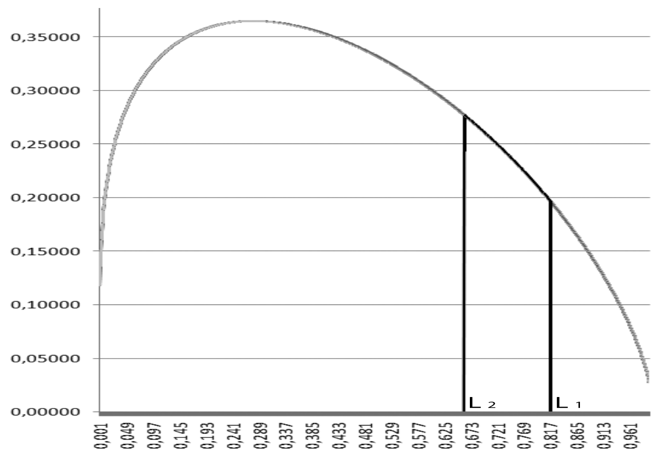

Figure 8: $\alpha = 0.4$, $\kappa = 0.5$.

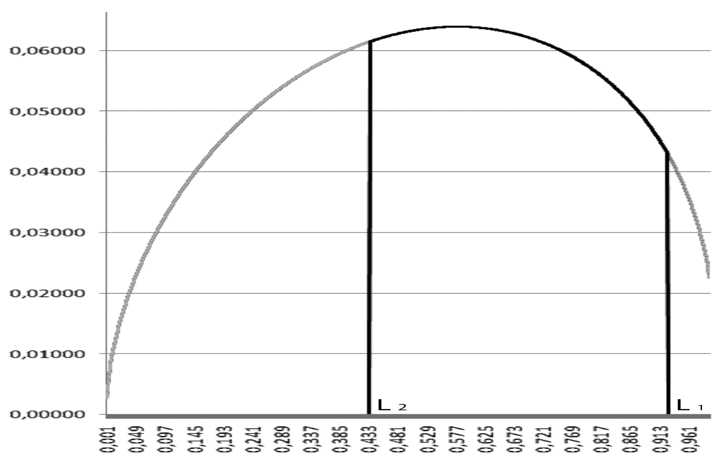

Figure 9: $\alpha = 0.7$, $\kappa = 0.8$.

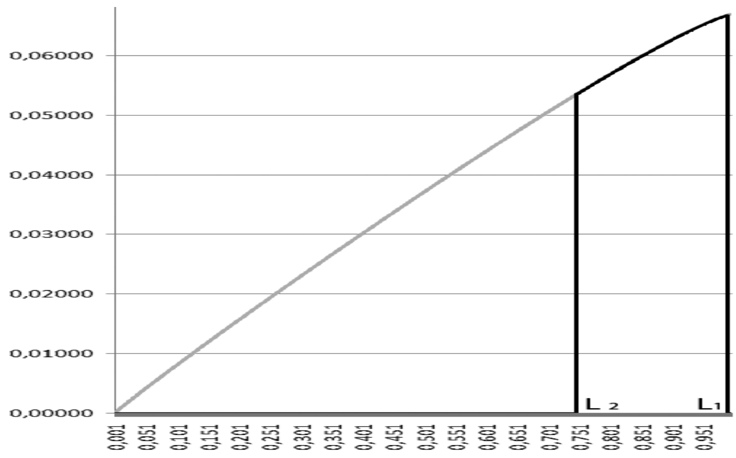

Figure 10: $\alpha = 0.3$, $\kappa = 0.96$.

8 Conclusion

To contribute to the analysis of the relationship between inequality and growth we developed a model characterized by (i) human capital accumulation, (ii) a family altruism motive, and (iii) a twofold occupational choice between skilled and unskilled positions.

We showed how the initial income distribution evolves across time both between skilled and unskilled agents and among skilled agents. The growth process contributes to increase the share of skilled workers on the transition. However in the long run, income inequality between skilled and unskilled workers persits while the inequality among skilled workers disappear.

The steady state equilibrium itself is actually a continuum and for each of these steady state a different wage inequality and splitting prevails. We show that the relationship between the location of the steady state pivotal dynasty, and hence the long run wage inequality, and the level of output (or the rate of growth of output) is ambiguous.

One possible extension of this approach could be the analysis of the respective merits of redistributive and educational policies in dealing with the long run relationship between inequality and growth. How would these policies change the growth process and the inequality between skilled and unskilled workers? And how would inequality among skilled agents be affected? This model provides a tool to jointly answer these two related questions.

Finally, another extension would consist in relaxing the closed economy assumption and to assume that, in a North-south framework globalization increases the supply of unskilled labor intensive goods. This is likely to change the results, especially the dynamics of inequality inside the northern countries. Indeed the effects of the increase in the share of skilled workers displayed by our closed-economy model would be couteracted by the globalization process.

References

[1] Aghion, P., E. Caroli and C. Garcia-Penalosa (1999), Inequality and economic growth: The perspective of new growth theories, Journal of Economic Literature, Vol XXXVII (December 1999), pp. 1615-1660

[2] Alesina, A. and Perotti, R. (1996), Income distribution, political instability, and investment. European Economic Review, 40, 1203-1228.

[3] Alesina, A. and Rodrik, A. (1994), Distributive politics and economic growth. Quarterly Journal of Economics, 109, 465-490.

[4] Andreoni, J., (1989), Giving with impure altruism: applications to charity and Ricardian equivalence. Journal of Political Economy 96, 1447–1458.

[5] Banerjee, A. and A. Newman, (1993) 'Occupational choice and the process of development, Journal of Political Economy 101, pp 274-298

[6] Barro, R.J.(1974). Are government bonds net wealth? Journal of Political Economy 82, 1095–1117.

[7] Becker G. (1964), Human capital. A theoretical and empirical analysis, with special reference to education, First Edition, NBER, New York.

[8] Becker, G. S. and N. Tomes, (1979), An Equilibrium Theory of the Distribution of Income and Intergenerational Mobility, The Journal of Political Economy, Vol. 87, N6 (Dec., 1979), 1153-1189

[9] Bertola, G. (1993), Factor shares and savings in endogenous growth, American Economic Review, 83, 1184-1198

[10] Borissov, K. and S. Lambrecht (2009), Growth and Distribution in an AK-model with Endogenous Impatience, 39 (1), Economic Theory pp. 93-112

[11] Chusseau, N. and J. Hellier, Educational systems, Intergenerational Mobility and Segmentation, MONDES Working Paper (abstract on http://inequality-globalisation.univ-lille1.fr/papers.php)

[12] Freeman, S., (1996) Equilibrium income inequality among identical agents, Journal of Political Economy , 104(4), pp 1047-1064

[13] Galor, O., Zeira, J., (1993), Income distribution and macroeconomics. Review of Economic Studies, 60, 35-52.

[14] Glomm G. and B. Ravikumar, (1992), Public versus Private Investment in Human Capital: Endogenous Growth and Income Inequality, Journal of Political Economy, vol 100, n41, pp 818-834.

[15] Grandmont, J.-M., (1983), Money and value: a reconsideration of classical and neoclassical monetary theories(Econometric Society of Monographs in Pure Theory, Number 5). Cambridge, New York and Melbourne: Cambridge University Press.

[16] Hicks, J., (1939), Value and capital, Oxford: Clarendon press,

[17] Lambrecht S., Michel Ph., Vidal J-P., (2005), Public pensions and growth. European Economic Review; 49 (5); 1261-1281.

[18] Lambrecht, S., Michel Ph., Thibault E., (2006), Capital accumulation and fiscal policy in an OLG model with family altruism. Journal of Public Economic Theory 8(3); 465-486.

[19] Loury, G.C., (1981), Intergenerational transfers and the distribution of earnings, Econometrica, vol. 49, n4, 843-867

[20] Lucas R.E. Jr. (1988), On the Mechanisms of Economic Development, Journal of Monetary Economics, 22, pp. 3-42.

[21] Mookherjee, D. and D. Ray, Inequality and Markets: Some Implications of Occupational Diversity, American Economic Journal, forthcoming

[22] Persson, T. and Tabellini, G., (1994), Is inequality harmful for growth? Theory and evidence. American Economic Review, 84, 600-621.

[23] Ray, D., (2006), On the Dynamics of Inequality. Economic Theory, 29, 291-306

[24] Schneider A., 2010, Redistributive taxation vs. education subsidies: Fostering equality and social mobility in an intergenerational model, Economics of Education review, 29, pp 597-605

A Proof of Proposition 1

The dynasty i time t individual problem, namely

$$\max_{c_t(i)\geq 0, e_t(i)\geq 0} \ln c_t(i) + \ln \omega_{t+1}(i),$$

under the following constraints:

$$e_t(i) + c_t(i) = \omega_t(i), \tag{34}$$

$$\omega_{t+1}(i) = \max\{w^L_{t+1}, w^H_{t+1} h_{t+1}(i)\}, \tag{35}$$

$$h_{t+1}(i) = e_t(i)^\kappa (h_t(i) + 1)^{1-\kappa}, \tag{36}$$

can be rewritten as follows:

$$\max_{0\leq e_t(i)\leq \omega_t(i)} \ln(\omega_t(i) - e_t(i)) + \ln(\max\{w^L_{t+1}, w^H_{t+1} e_t(i)^\kappa (h_t(i) + 1)^{1-\kappa}\}). \tag{37}$$

This problem can be solved in two steps. First we solve the two sub-problems defined by the two alternatives of the max function,

$$\max_{0\leq e_t(i)\leq \omega_t(i)} \ln(\omega_t(i) - e_t(i)) + \ln(w^L_{t+1}), \tag{38}$$

and

$$\max_{0\leq e_t(i)\leq \omega_t(i)} \ln(\omega_t(i) - e_t(i)) + \ln(w^H_{t+1} e_t(i)^\kappa (h_t(i) + 1)^{1-\kappa}), \tag{39}$$

and then we select the solution leading to the highest utility.

The solution to problem (38) is $e_t(i) = 0$. If at time $t + 1$ dynasty i is going to supply unskilled labor on the labor market, then at time t its expenditure on education is nil and all income $\omega_t(i)$ is spent on consumption. Hence the human capital of this dynasty at time $t+1$, $h_{t+1}(i)$, is also nil. The value of problem (38) is

$$V^L\big(\omega_t(i), h_t(i)\big) = \ln \omega_t(i) + \ln w_{t+1}^L.$$

The solution to problem (39) is

$$e_t(i) = \hat{e}^H\big(\omega_t(i), h_t(i)\big) = \frac{\kappa}{1+\kappa}\omega_t(i).$$

If at time $t+1$ dynasty i is going to supply human capital on the labor market, then at time t its expenditure on education is equal to the fraction $\frac{\kappa}{1+\kappa}$ of its income $\omega_t(i)$. The rest of the income, $\frac{1}{1+\kappa}\omega_t(i)$, is spent on consumption. Thus, the endowment of human capital of agent that belongs to dynast i and lives at time $t+1$ is equal to

$$\hat{h}^H\big(\omega_t(i), h_t(i)\big) = \gamma\big(\omega_t(i)\big)^\kappa\big(1 + h_t(i)\big)^{1-\kappa}, \tag{40}$$

where

$$\gamma = \left(\frac{\kappa}{1+\kappa}\right)^\kappa.$$

The value of problem (39) is

$$V^H\big(\omega_t(i), h_t(i)\big) = \ln \omega_t(i) - \ln(1+\kappa) + \ln w_{t+1}^H + \ln \hat{h}^H\big(\omega_t(i), h_t(i)\big).$$

It is clear that if $V^H\big(\omega_t(i), h_t(i)\big) < V^L\big(\omega_t(i), h_t(i)\big)$, then the unique solution to problem (37) coincides with the solution to problem (38). If $V^H\big(\omega_t(i), h_t(i)\big) > V^L\big(\omega_t(i), h_t(i)\big)$, then the unique solution to problem (37) coincides with the solution to problem (39), and if $V^H\big(\omega_t(i), h_t(i)\big) = V^L\big(\omega_t(i), h_t(i)\big)$, then the solutions of both (38) and (39) are solutions to (37). It is easily checked that

$$V^H\big(\omega_t(i), h_t(i)\big) \gtreqqless V^L\big(\omega_t(i), h_t(i)\big) \;\Leftrightarrow\; \frac{\hat{h}^H\big(\omega_t(i), h_t(i)\big)}{1+\kappa} \gtreqqless \frac{w_{t+1}^L}{w_{t+1}^H}.$$

B Proof of Lemma 2

First we show that on the set of steady-state equilibria there is a monotonically decreasing relationship between the equilibrium pivotal dynasty i^{H*}, which shows the share of uneducated agents in the population, and the amount of human capital, h^*, supplied by each educated agent. Second, we show that the set of equilibrium values of i^H is an interval.

Let us first prove the following lemma.

Lemma 6. 1) *There is a smooth increasing function $\chi : (0,1) \to \mathbb{R}_+$ such that for any $i \in (0,1)$,*

$$h = \gamma\left[\alpha\left(\frac{i}{(1-i)h}\right)^{1-\alpha}\right]^\kappa \psi(h) \Leftrightarrow h = \chi(i). \tag{41}$$

This function satisfies

$$\chi(i) \to 0 \text{ as } i \to 0 \qquad \text{and} \qquad \chi(i) \to \infty \text{ as } i \to 1. \tag{42}$$

2) $\frac{1-i}{i}\chi(i)$ *monotonically decreases from ∞ to $(\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}}$ as i increases from 0 to 1.*

Proof. 1) Given that $\psi(h) = h^\kappa(h+1)^{1-\kappa}$, we can rewrite the first equation in (41) as

$$h = \gamma\alpha^\kappa \left(\frac{i}{1-i}\right)^{(1-\alpha)\kappa} h^{\alpha\kappa}(h+1)^{1-\kappa},$$

or, after dividing both sides by $h^{\alpha\kappa}(h+1)^{1-\kappa}$, as

$$\frac{h^{1-\alpha\kappa}}{(h+1)^{1-\kappa}} = \gamma\alpha^\kappa \left(\frac{i}{1-i}\right)^{(1-\alpha)\kappa}. \tag{43}$$

The LHS of the last equation is continuous and increasing in h, tends to 0 as $h \to 0$ and to $+\infty$ as h tends to $+\infty$ since $1-\alpha\kappa > 1-\kappa$. The RHS of this equation is continuous and increasing in i, tends to 0 as $i \to 0$ and to $+\infty$ as $i \to 1$. It is then obvious that for any i there exists a solution to (43) in h. To complete the proof, denote this solution by $\chi(i)$ and notice that the both properties in (42) hold true.

2) After some rearrangement of (43) we can get

$$\frac{1-i}{i}\chi(i) = (\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}} \left(1 + \frac{1}{\chi(i)}\right)^{(1-\kappa)/(1-\alpha)\kappa}.$$

Since $\chi(i)$ monotonically increases from 0 to ∞ , $\frac{1-i}{i}\chi(i)$ monotonically decreases from ∞ to $(\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}}$ as i increases from 0 to 1. $\square$

Because of (25) and Lemma 6, equation (30) can be rewritten as (31), where $\chi(\cdot)$ is the function introduced in Lemma 6. Thus, the amount of human capital supplied by each educated agent is an increasing function of the share of uneducated agents in the population.

Also, by (27) and (30), we can rewrite (29) as

$$\frac{h^*}{1+\kappa} \geq \frac{1-\alpha}{\alpha}\frac{(1-i^{H*})h^*}{i^{H*}},$$

or, equivalently, as

$$i^{H*} \geq L_2 = \frac{(1-\alpha)(1+\kappa)}{(1-\alpha)(1+\kappa)+\alpha}.$$

Let us now rewrite (28) as

$$\frac{\gamma}{1+\kappa}(1-\alpha)^\kappa \left(\frac{(1-i^{H*})h^*}{i^{H*}}\right)^{\alpha\kappa} \leq \frac{1-\alpha}{\alpha}\frac{(1-i^{H*})h^*}{i^{H*}},$$

This inequality can be re-written as

$$\frac{\gamma}{1+\kappa}(1-\alpha)^\kappa \leq \frac{1-\alpha}{\alpha}\left(\frac{1-i^{H*}}{i^{H*}}h^*\right)^{1-\alpha\kappa} = \frac{1-\alpha}{\alpha}\left(\frac{1-i^{H*}}{i^{H*}}\chi(i^{H*})\right)^{1-\alpha\kappa}$$

or, after substituting (31) as

$$\frac{\gamma}{1+\kappa}(1-\alpha)^\kappa \leq \frac{1-\alpha}{\alpha}\left(\frac{1-i^{H*}}{i^{H*}}\chi(i^{H*})\right)^{1-\alpha\kappa}. \tag{44}$$

By Lemma 6, $\frac{1-i}{i}\chi(i)$ monotonically decreases from ∞ to $(\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}}$ as i increases from 0 to 1. Therefore, if

$$\left(\frac{\alpha\gamma}{1+\kappa}(1-\alpha)^{\kappa-1}\right)^{\frac{1}{1-\alpha\kappa}} \geq (\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}},$$

then (44) is equivalent to

$$i^{H*} \leq L_1, \tag{45}$$

where L_1 is the solution to the following equation in i:

$$\frac{\gamma}{1+\kappa}(1-\alpha)^\kappa = \frac{1-\alpha}{\alpha}\left(\frac{1-i}{i}\chi(i)\right)^{1-\alpha\kappa}.$$

If

$$\left(\frac{\alpha\gamma}{1+\kappa}(1-\alpha)^{\kappa-1}\right)^{\frac{1}{1-\alpha\kappa}} < (\gamma\alpha^\kappa)^{\frac{1}{(1-\alpha)\kappa}},$$

then (44) holds for all $i^{H*} \in (0,1)$.

To complete the the proof of Lemma 2, it is necessary to show that $L_2 < L_1$. To do this, let

$$h^{**} = \chi(L_2),$$

$$w^{H**} = \alpha\left(\frac{L_2}{(1-L_2)h^{**}}\right)^{1-\alpha},\ w^{L**} = (1-\alpha)\left(\frac{(1-L_2)h^{**}}{L_2}\right)^{\alpha}.$$

We have

$$\frac{h^{**}}{1+\kappa} = \frac{\gamma(w^{H**})^\kappa\psi(h^{**})}{1+\kappa} = \frac{w^{L**}}{w^{H**}}.$$

It follows that

$$w^{H**}h^{**} > \frac{w^{H**}h^{**}}{1+\kappa} = w^{L**}.$$

Hence

$$\frac{\gamma(w^{L**})^\kappa}{1+\kappa} < \frac{\gamma(w^{H**}h^{**})^\kappa}{1+\kappa} < \frac{\gamma(w^{H**})^\kappa(h^{**})^\kappa(h^{**}+1)^{1-\kappa}}{1+\kappa}$$
$$= \frac{\gamma(w^{H**})^\kappa\psi(h^{**})}{1+\kappa} = \frac{w^{L**}}{w^{H**}}.$$

It is clear that (28) fulfills as a strict inequality if and only if (45) fulfills as a strict inequality. Therefore the last chain on inequalities implies $L_2 < L_1$. $\square$

C Proof of Theorem 4

It is sufficient to prove that for any equilibrium path $\{i_t^H, h_{t+1}(\cdot)\}_{t=0}^\infty$ the sequence $\{i_t^H\}_{t=1}^\infty$ is non-increasing.

Let $\{i_t^H, h_{t+1}(\cdot)\}_{t=0}^\infty$ be an equilibrium path. Show that

$$i_{\tau+1}^H \le i_\tau^H, \ \tau = 1, 2, \dots. \tag{46}$$

Let us take a $\tau > 0$. It follows from (16) that

$$\mathcal{H}_{\tau+1}(i_\tau^H) = \gamma \left(w_\tau^H\right)^\kappa \int_{i_\tau^H}^1 \psi(h_\tau(i))di, \tag{47}$$

(the functions $\psi(\cdot)$ and $\mathcal{H}_{\tau+1}(\cdot)$ are defined by respectively (17) and (18)).

Since $\frac{\psi(h)}{h} = \left(\frac{h+1}{h}\right)^{1-\kappa}$ is a decreasing function of $h > 0$, we have

$$\frac{\psi[h_\tau(i)]}{h_\tau(i)} \le \frac{\psi[h_\tau(i_\tau^H)]}{h_\tau(i_\tau^H)}, \ i_\tau^H \le i \le 1.$$

This inequality can be rewritten as

$$\psi[h_\tau(i)] \le \frac{\psi[h_\tau(i_\tau^H)]}{h_\tau(i_\tau^H)} h_\tau(i), \ i_\tau^H \le i \le 1.$$

Taking account of (47) and (24), we get

$$\mathcal{H}_{\tau+1}(i_\tau^H) \le \gamma \left(w_\tau^H\right)^\kappa \frac{\psi[h_\tau(i_\tau^H)]}{h_\tau(i_\tau^H)} \int_{i_\tau^H}^1 h_\tau(i)di$$

$$= \gamma \left(w_\tau^H\right)^\kappa \frac{\psi[h_\tau(i_\tau^H)]}{h_\tau(i_\tau^H)} \int_{i_\tau^H}^1 \tilde{h}_\tau(i)di = \tilde{h}_{\tau+1}(i_\tau^H) \frac{\mathcal{H}_\tau(i_\tau^H)}{\tilde{h}_\tau(i_\tau^H)}$$

and hence

$$\frac{\mathcal{H}_{\tau+1}(i_\tau^H)}{\tilde{h}_{\tau+1}(i_\tau^H)} \le \frac{\mathcal{H}_\tau(i_\tau^H)}{\tilde{h}_\tau(i_\tau^H)}.$$

Suppose now that $i_\tau^H < i_{\tau+1}^H$. It follows from (22) applied to $t = \tau$ and $j = i_\tau^H$ that

$$\frac{\alpha}{1-\alpha} \frac{1}{1+\kappa} i_\tau^H < \frac{\mathcal{H}_{\tau+1}(i_\tau^H)}{\tilde{h}_{\tau+1}(i_\tau^H)}.$$

At the same time, from (23) applied to $t = \tau - 1$ and $j = i_\tau^H$,

$$\frac{\mathcal{H}_\tau(i_\tau^H)}{\tilde{h}_\tau(i_\tau^H)} = \frac{\alpha}{1-\alpha} \frac{1}{1+\kappa} i_\tau^H.$$

The last three inequalities imply $i_\tau^H > i_\tau^H$, which is a contradiction. This contradiction proves (46).

To complete the proof, it is sufficient to note that the sequence $\{i_t^H\}_{t=1}^\infty$ is bounded from below and therefore converges to some i^{H*}. It is no difficult to check that $i^{H*} > 0$ and the required steady-state equilibrium is the couple (i^{H*}, h^*), where $h^* = \chi(i^{H*})$. $\square$

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